Elasticity of Residual Supply

From the lecture slides, you should know that a buyer faces a residual supply curve: the quantity the market supplies that is not consumed by other demanders at any given price. Formally, this is given by:

$$S^r(p) = S(p) - D_0(p)$$

This is the difference between the market supply $S(p)$ and the quantity of the good demanded by the other buyers $D_0(p)$. How do we derive the formula found in the lecture slides?

Let's start by recalling the formula for residual elasticity for an individual buyer:

$$\varepsilon_s^r = \frac{dS^r}{dp}\frac{p}{S^r}$$

Hence, we can differentiate both sides of the residual supply equation above and multiple both sides by $\frac{p}{S^r}$ to obtain:

$$\frac{dS^r}{dp}\frac{p}{S^r} = \frac{dS}{dp}\frac{p}{S^r} - \frac{dD_0}{dp}\frac{p}{S^r}$$

This, by definition, simplifies to:

$$\varepsilon_s^r = \frac{dS}{dp}\frac{p}{S^r} - \frac{dD_0}{dp}\frac{p}{S^r}$$

Now we can multiply the first term by $\frac{S}{S}$. Since this equals one, it does not change anything in the equation. We thus obtain:

$$\varepsilon_s^r = \frac{dS}{dp}\frac{S}{S}\frac{p}{S^r} - \frac{dD_0}{dp}\frac{p}{S^r}$$

Rearranging, we get:

$$\varepsilon_s^r = \frac{dS}{dp}\frac{p}{S}\frac{S}{S^r} - \frac{dD_0}{dp}\frac{p}{S^r}$$

Now notice that $\frac{S}{S^r}$ is the reciprocal of the buyer's share, while $\frac{dS}{dp}\frac{p}{S}$ is the market elasticity of supply. Hence, we can substitute these in to get:

$$\varepsilon_s^r = \frac{\eta}{\theta} - \frac{dD_0}{dp}\frac{p}{S^r}$$

For the second term, we multiply it by $\frac{D_0}{D_0}$. Again since this equal 1 it changes nothing in the equation but allows us to simplify things, and through rearranging (and applying the residual demand elasticity formula) we obtain:

$$\varepsilon_s^r = \frac{\eta}{\theta} - \frac{dD_0}{dp}\frac{D_0}{D_0}\frac{p}{S^r} = \frac{\eta}{\theta} - \frac{dD_0}{dp}\frac{p}{D_0}\frac{D_0}{S^r} = \boldsymbol{\frac{\eta}{\theta} - \varepsilon_0\frac{D_0}{S^r}}$$

Lastly, we note that via the definition of residual supply, $D_0 = S - S^r$ (I have dropped the (p) for simplicity of notation). This yields:

$$\varepsilon_s^r = \frac{\eta}{\theta} - \varepsilon_0 \frac{S - S^r}{S^r}$$

To get the formula from your lecture slides, we now have to multiply the second term by $\frac{1/S}{1/S}$. Notice again that this is equal to 1 (though looks a bit weird). This gives us:

$$\varepsilon_s^r = \frac{\eta}{\theta} - \varepsilon_0 \frac{(S - S^r)/S}{S^r/S} = \frac{\eta}{\theta} - \varepsilon_0 \frac{1 - \theta}{\theta}$$

This final result just is the formula that you have been provided:

$$\varepsilon_s^r = \frac{\eta}{\theta} - \frac{1 - \theta}{\theta} \varepsilon_0$$

Obviously, this is a derivation that you do not need to learn for your exam, but it shows where the formula actually comes from!