

Cournot Equilibrium Formula Derivation

Assumptions

1. n firms in the market
2. $p = a - bQ = a - b(q_1 + q_2 + \dots + q_n) = a - b(\sum_{i=1}^n q_i)$
3. $MC_i = m$ ($\forall i = 1, \dots, n$)

Derivations

$$TR_i = pq_i = [a - b(q_1 + q_2 + \dots + q_n)]q_i \quad (\text{derive total revenue})$$

$$MR_i = \frac{dTR_i}{dq_i} = a - b(q_1 + q_2 + \dots + q_n + q_i) \quad (\text{derive marginal revenue})$$

$$MR_i = MC \Leftrightarrow a - b(q_1 + q_2 + \dots + q_n + q_i) = m \quad (\text{profit maximisation})$$

Suppose now that we select two firms i and j , and find the difference between their reaction functions:

$$(a - b(q_1 + q_2 + \dots + q_n + q_i)) - (a - b(q_1 + q_2 + \dots + q_n + q_j)) = 0 \quad (\text{by elimination})$$
$$-bq_i + bq_j = 0 \Rightarrow q_i = q_j \quad (\text{simplifying})$$

This implies that $q_1 = \dots = q_n$. Hence, all firms are producing the same quantity in equilibrium. This allows us to simplify the reaction function:

$$a - b(q_i + q_i + \dots + q_i + q_i) = m \quad (\text{by substitution})$$

$$a - b(n + 1)q_i = m \quad (\text{by factorisation})$$

$$q_i^* = \frac{a - m}{b(n + 1)}$$

To derive the rule for market price, we simplify the market inverse demand function:

$$p = a - b(q_1 + q_2 + \dots + q_n) = a - bnq_i \quad (\text{by substitution})$$

$$p = a - bn \frac{a - m}{b(n + 1)} = a + \frac{-an + mn}{(n + 1)} = \frac{a(n + 1) - an + mn}{n + 1} \quad (\text{by simplification})$$

$$p^* = \frac{a + mn}{(n + 1)}$$

Stackelberg Duopoly Equilibrium Derivation

Assumptions

1. $i = 1, 2$ where firm 1 is the leader, and firm 2 is the follower.
2. $p = a - b(q_1 + q_2)$ (inverse demand function is a function of total production)
3. $MC_i = c$ ($\forall i = 1, \dots, n$)

Derivation

We use backward induction to solve for the Nash equilibrium. Since firm 2 is the follower, we find the optimal reaction function for it first:

$$\begin{aligned}\pi_2 &= q_2 p - TC_2 \\ \frac{\partial \pi_2}{\partial q_2} &= p + q_2 \frac{\partial p}{\partial q_2} - c = 0 \\ a - bq_1 - bq_2 - c &= 0 \\ q_2^* &= \frac{a - bq_1 - c}{2b}\end{aligned}$$

This reaction function is the same as in the Cournot equilibrium case. However, since this game is sequential, firm 1 will take this reaction function into account when making a decision:

$$\begin{aligned}\pi_1 &= q_1 p - TC_1 \\ \frac{\partial \pi_1}{\partial q_1} &= p + q_1 \left(\frac{\partial p}{\partial q_1} + \frac{\partial p}{\partial q_2} \frac{\partial q_2}{\partial q_1} \right) - c = 0\end{aligned}$$

This may look complicated, but another way of seeing this is that from firm 1's perspective the inverse demand function now looks like this:

$$p = a - b(q_1 + q_2); q_2^* = \frac{a - bq_1 - c}{2b} \Rightarrow p = a - b \left(q_1 + \frac{a - bq_1 - c}{2b} \right)$$

Hence, we can differentiate to obtain:

$$\begin{aligned}a - b \left(q_1 + \frac{a - bq_1 - c}{2b} \right) + \left(-b + \frac{b}{2} \right) q_1 - c &= 0 \\ a - bq_1 - \left(\frac{a - c}{2} \right) - c &= 0 \\ q_1^* &= \frac{a - c}{2b} \\ q_2^* &= \frac{a - bq_1 - c}{2b} = \frac{a - b \left(\frac{a - c}{2b} \right) - c}{2b} = \frac{a - c}{4b}\end{aligned}$$

As you can see, the leader firm has a **first mover advantage** and has a higher market share in the market, and so earns more profit!

Lerner Index Formula Derivation

Assumptions

1. $Q = q_1 + q_2 + \dots + q_n = \sum_{i=1}^n q_i$ (total production is just the sum of individual production)
2. $p = f(Q)$ (inverse demand function is a function of total production)
3. $MC_i = c$ ($\forall i = 1, \dots, n$)

Mathematical Note

$$\varepsilon = \frac{dQ}{dp} \frac{p}{Q} \Rightarrow \frac{1}{\varepsilon} = \frac{dp}{dQ} \frac{Q}{p}$$

Derivation

The profits of an individual firm are given by:

$$\pi_i = pq_i - TC_i$$

$$\frac{\partial \pi_i}{\partial q_i} = p + q_i \frac{\partial p}{\partial q_i} - c = 0$$

$$p \left(1 + \frac{q_i}{p} \frac{\partial p}{\partial q_i} \right) - c = 0$$

$$p \left(1 + \frac{q_i Q}{Q p} \frac{\partial p}{\partial q_i} \right) - c = 0$$

$$p \left(1 + \frac{q_i}{Q} \frac{1}{\varepsilon} \right) - c = 0$$

$$p \left(1 + \frac{1}{n\varepsilon} \right) - c = 0$$

$$p - c = -\frac{p}{n\varepsilon}$$

$$\frac{p - c}{p} = -\frac{1}{n\varepsilon}$$

Thus, we arrive at our final result called the **Lerner Index** of product market competition.