Externalities and Equilibrium - SOLUTIONS

Note: you will see that the notation for some of the general equilibrium questions varies. This is deliberate. It is an important skill to be able to adapt to different types of notation and show greater understanding!

Practice Problems

1. Work out the consumer surplus or producer surplus (depending on whether supply or demand are given) for the following:

   a. $p = 15 - Q_d, p^e = 10$

$$CS = 0.5 \times (15 - 10) \times 5 = \mathbf{12.5}$$
$$CS = \int_0^5 (15 - Q)dQ - 50 = \mathbf{12.5}$$

   b. $p = 5 + 2Q_s, Q^e = 4$

$$PS = 0.5 \times (13 - 5) \times 4 = \mathbf{16}$$
$$PS = 52 - \int_0^4 (5 + 2Q)dQ = \mathbf{16}$$

   For those confused about this. Think of it geometrically: which triangle area are we looking for in the diagram?

2. Consider an economy with two goods and two agents, Elisa, and Bob. Elisa is initially endowed with $E = (5, 30)$ and Bob's endowment is $B = (20,20)$. Elisa and Ben have the same utility given by $U(x_1, x_2) = 2\ln x_1 + 2\ln x_2$.

   a. What is the marginal rate of substitution of each individual?

   This is just a transformed version of the Cobb-Doughlas function. Hence, the MRS for each consumer is:

$$\boldsymbol{MRS_E = \frac{x_2^E}{x_1^E};\ MRS_B = \frac{x_2^B}{x_1^B}}$$

   b. Derive the contract curve for this economy.

   The aggregate endowments imply that:

$$x_1^E + x_1^B = 20 + 5 = 25$$
$$x_2^E + x_2^B = 30 + 20 = 50$$

   Hence, we can write $x_1^B = 25 - x_1^E$ and $x_2^B = 50 - x_2^E$. At the Pareto equilibrium:

$$MRS_E = \frac{x_2^E}{x_1^E} = MRS_B = \frac{x_2^B}{x_1^B}$$

Hence, we can write:

$$x_2^E x_1^B = x_1^E x_2^B$$

By substitution from above:

$$x_2^E(25 - x_1^E) = x_1^E(50 - x_2^E)$$

$$25x_2^E = 50x_1^E \Longrightarrow \boldsymbol{x_2^E = 2x_1^E}$$

3. Consider an economy with two goods and two agents, Elisa, and Bob. Elisa is initially endowed with $E = (10, 20)$ and Bob's endowment is $B = (40,40)$. Elisa and Bob have utility functions given by $U_E(x, y) = 5x_E^{0.5}y_E^{0.5}$ and $U_B(x, y) = 2x_B^{0.1}y_B^{0.8}$.

   a. What is the marginal rate of substitution of each individual?

$$\boldsymbol{MRS_E = \frac{y_E}{x_E};\ MRS_B = \frac{y_B}{8x_B}}$$

   b. Derive the contract curve for this economy.

   The aggregate endowments are:

$$y_E + y_B = 50$$

$$x_E + x_B = 60$$

   The Pareto equilibrium occurs only if:

$$\frac{y_E}{x_E} = \frac{y_B}{8x_B}$$

   Rearranging:

$$8y_E x_B = y_B x_E$$

   Using the aggregate endowment constraints:

$$8y_E(60 - x_E) = (50 - y_E)x_E$$

$$480y_E - 7y_E x_E = 50x_E$$

$$\boldsymbol{y_E = \frac{50x_E}{480 - 7x_E}}$$

4. Suppose that consumer A has a utility function of $U_A(x,y) = \min\{x_A, y_A\}$ while consumer B has the utility function $U_B(x,y) = x_B y_B$. Suppose that their respective endowments are $A = (1,2)$ and $B = (1,2)$.

   a. If a benevolent central planner reassigns the endowments such that now $A = (1,1)$ and $B = (2,2)$. Is this new allocation a Pareto improvement over the original one? Is it Pareto optimal, that is, can we find another allocation that can make at least one consumer better off and no one worse off)?

      It is Pareto improvement. This is because $U_A^0 = \min\{2,1\} = 1, U_B^0 = 1 \times 2 = 2$, while, after the reassignment, $U_A^1 = \min\{1,1\} = 1, U_B^0 = 2 \times 2 = 4$. Hence, consumer A has the same utility as before, while consumer B has experienced an increase in utility. Indeed, this new allocation is a Pareto optimal. Since the marginal utility of both goods is positive for consumer 2, we could not reassign any goods from them without reducing their utility.

   b. Can a central planner achieve an allocation such that $A = (2,2)$ and $B = (2,2)$? If not, why not?

      No, because it exceeds the original endowments. Since this is an exchange economy where neither good is produced, an allocation that exceeds the aggregate endowments of the economy is impossible.

   c. Suppose we made consumer A altruistic such that their new utility function is now $U_A(x,y) = \min\{4x_A, y_B\}$ and now depends on the consumption of good y by consumer B. What would be the Pareto optimal allocation now?

      The Pareto optimal allocation is $A = (1,0)$ and $B = (1,4)$ yielding 4 utility for both consumers. Notice again that there is no Pareto improvement that can be made from this assignment.

5. Work out the consumer and producer surplus for the following equilibria:

   a. $p = 18 - 0.5Q; p = \frac{1}{3}Q + 3$

$$\boldsymbol{p = 9, Q = 18}$$

$$CS = 0.5 \times (18 - 9) \times 18 = \mathbf{81}$$

$$CS = \int_0^{18} (18 - 0.5Q)dQ - 162 = 81$$

$$PS = 0.5 \times (9 - 3) \times 18 = \mathbf{54}$$

$$PS = 162 - \int_0^{18} \left(\frac{1}{3}Q + 3\right) dQ = \mathbf{54}$$

b. $P = 240 - 6Q; P = 120 + 4Q$

$$\boldsymbol{p = 168, Q = 12}$$

$$CS = 0.5 \times (240 - 168) \times 12 = \mathbf{432}$$
$$CS = \int_0^{12} (240 - 6Q)dQ - 2016 = \mathbf{432}$$

$$PS = 0.5 \times (168 - 120) \times 12 = \mathbf{288}$$
$$PS = 2016 - \int_0^{12} (120 + 4Q)dQ = \mathbf{288}$$

6. Now suppose that for both equilibria considered in the previous question, the government introduces a £2 per unit tax. Calculate the new consumer surplus, producer surplus as well as the deadweight loss to society from this tax.

a. After-tax equations: $p = 18 - 0.5Q; p = \frac{1}{3}Q + 5$

$$\boldsymbol{p = 10.2, Q = 15.6}$$

$$CS = 0.5 \times (18 - 10.2) \times 15.6 = \mathbf{60.84}$$
$$CS = \int_0^{15.6} (18 - 0.5Q)dQ - 159.12 = \mathbf{60.84}$$

$$PS = 0.5 \times (10.2 - 5) \times 15.6 = \mathbf{40.56}$$
$$PS = 159.12 - \int_0^{15.6} \left(\frac{1}{3}Q + 5\right) dQ = \mathbf{40.56}$$
$$g_\tau = 2 \times 15.6 = \mathbf{31.2}$$

Deadweight loss easily be found geometrically.

$$DWL = \frac{1}{2} \times (18 - 15.6) \times \left(10.2 - \left(\frac{1}{3}(15.6) + 3\right)\right) = \mathbf{2.4}$$

With integrals, we must be a little more creative:

$$DWL = \int_{15.6}^{18} (18 - 0.5Q)\ dQ - \int_{15.6}^{18} \left(\frac{1}{3}Q + 3\right)\ dQ = \mathbf{2.4}$$

b. After-tax equations: $P = 240 - 6Q; P = 122 + 4Q$

$$\boldsymbol{p = 169.2, Q = 11.8}$$

$$CS = 0.5 \times (240 - 169.2) \times 11.8 = \mathbf{417.72}$$

$$CS = \int_0^{11.8} (240 - 6Q)dQ - 1996.56 = \mathbf{417.72}$$

$$PS = 0.5 \times (169.2 - 122) \times 11.8 = \mathbf{278.48}$$

$$PS = 1996.56 - \int_0^{11.8} (122 + 4Q)dQ = \mathbf{278.48}$$

$$g_\tau = 2 \times 11.8 = \mathbf{23.6}$$

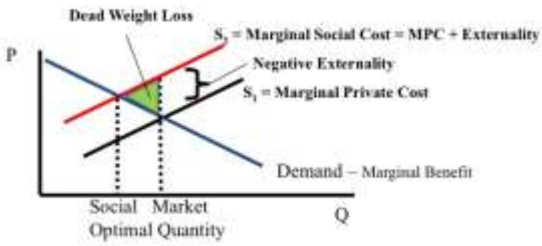
Deadweight loss easily be found geometrically.

$$DWL = \frac{1}{2} \times (12 - 11.8) \times \left(169.2 - \left(120 + 4(11.8)\right)\right) = \mathbf{0.2}$$

With integrals, we must be a little more creative:

$$DWL = \int_{11.8}^{12} (240 - 6Q)\ dQ - \int_{11.8}^{12} (120 + 4Q)\ dQ = \mathbf{0.2}$$

7. Market supply in a competitive industry is $p = Q + 10$, while demand is $p = 50 - 2Q$. Suppose that the production process imposes a social cost of \$3 per unit of output. The government attempts to correct this problem by imposing a \$2 tax on the producers. What effect does this policy have on the consumer surplus? What happens to the deadweight loss to society?



The original equilibrium is:

$$Q + 10 = 50 - 2Q \Rightarrow \boldsymbol{Q} = \frac{\mathbf{40}}{\mathbf{3}}, \boldsymbol{p} = \frac{\mathbf{70}}{\mathbf{3}}$$

Taking the social cost in account, we have that:

$$Q + 13 = 50 - 2Q \Rightarrow \boldsymbol{Q} = \frac{\mathbf{37}}{\mathbf{3}}, \boldsymbol{p} = \frac{\mathbf{76}}{\mathbf{3}}$$

$$CS = 0.5 \times \left(50 - \frac{76}{3}\right) \times \frac{37}{3} = \frac{\mathbf{1369}}{\mathbf{9}} = \mathbf{152.11}$$

$$CS = \int_0^{\frac{37}{3}} (50 - 2Q)dQ - \frac{2812}{9} = \frac{\mathbf{1369}}{\mathbf{9}} = \mathbf{152.11}$$

$$DWL = \frac{1}{2} \times \left(\frac{40}{3} - \frac{37}{3}\right) \times \left(\left(\frac{40}{3} + 13\right) - \frac{70}{3}\right) = \mathbf{1.5}$$

$$DWL = \int_{\frac{37}{3}}^{\frac{40}{3}} (Q + 13)\ dQ - \int_{\frac{37}{3}}^{\frac{40}{3}} (50 - 2Q)\ dQ = \mathbf{1.5}$$

The with-tax equilibrium is:

$$Q + 12 = 50 - 2Q \Rightarrow \boldsymbol{Q} = \frac{\mathbf{38}}{\mathbf{3}}, \boldsymbol{p} = \frac{\mathbf{74}}{\mathbf{3}}$$

$$CS = 0.5 \times \left(50 - \frac{74}{3}\right) \times \frac{38}{3} = \frac{\mathbf{1520}}{\mathbf{9}} = \mathbf{168.89}$$

$$CS = \int_0^{\frac{38}{3}} (50 - 2Q)dQ - \frac{2812}{9} = \frac{\mathbf{1444}}{\mathbf{9}} = \mathbf{160.44}$$

$$DWL = \frac{1}{2} \times \left(\frac{40}{3} - \frac{38}{3}\right) \times \left(\left(\frac{40}{3} + 12\right) - \frac{70}{3}\right) = \frac{\mathbf{2}}{\mathbf{3}}$$

$$DWL = \int_{\frac{38}{3}}^{\frac{40}{3}} (Q + 10)\ dQ - \int_{\frac{38}{3}}^{\frac{40}{3}} (50 - 2Q)\ dQ = \frac{\mathbf{2}}{\mathbf{3}}$$

Hence the tax policy reduces the deadweight loss and increases consumer surplus.

8. (***Easy Challenge***) Suppose that the inverse demand curve for a monopoly market is $P = 30 - 2Q$, while the marginal cost of the monopolist is simply $MC = 12$.

   a. Calculate the output and price in the monopoly market.

   $$MC = MR \Longrightarrow 12 = 30 - 4Q \Longrightarrow \boldsymbol{Q} = \mathbf{4.5}, \boldsymbol{P} = \mathbf{21}$$

   b. Calculate the equilibrium if this market was instead perfectly competitive.

   $$MC = P \Longrightarrow 12 = 30 - 2Q \Longrightarrow \boldsymbol{Q} = \mathbf{9}, \boldsymbol{P} = \mathbf{12}$$

   c. Find the consumer surplus for the cases in part (a) and part (b). Comment on how they compare to one another.

   Monopoly:

   $$CS = 0.5 \times (30 - 21) \times 4.5 = \mathbf{20.25}$$

   $$CS = \int_0^{4.5} (30 - 2Q)dQ - 94.5 = \mathbf{20.25}$$

Perfect Competition:

$$CS = 0.5 \times (30 - 12) \times 9 = \mathbf{81}$$

$$CS = \int_0^9 (30 - 2Q)dQ - 108 = \mathbf{81}$$

Not unexpectedly, the consumer surplus is greater under perfect competition that under a monopoly.