Game Theory – SOLUTIONS

Practice Problems

1. Using either the iterative elimination of dominated strategies or by identifying the best responses of each player, find the pure strategy Nash equilibrium of the following games.

a.

		Player 2	
		A	B
Player 1	**A**	**(1,1)**	**(20,0)**
	B	**(0,20)**	**(10,10)**

This should be recognised as a prisoners' dilemma game. Hence, the Nash equilibrium is to defect for both players: (1,1). This is a dominant strategy Nash equilibrium because A is a dominant strategy for both players.

b.

		Player 2	
		A	B
Player 1	**A**	**(3,2)**	**(6,4)**
	B	**(5,7)**	**(1,1)**

Use the best response approach to find this Nash equilibrium. Neither player has a dominant strategy here.

c.

		Player 2		
		A	B	C
Player 1	**A**	**(5,8)**	**(3,2)**	**(8,1)**
	B	**(2,2)**	**(6,9)**	**(7,1)**
	C	**(1,8)**	**(2,3)**	**(4,2)**

This looks like a difficult problem. However, it should immediately be recognised that C will never be played by either player. For both players, it yields the lowest payoff. It is called a strictly dominated strategy. You could also solve this using best responses. Either approach works.

2. By following the normal procedure when finding a mixed strategy equilibrium, show why in the following prisoners' dilemma game, no such equilibrium is possible:

		Player 2	
		A	B
Player 1	**A**	**(1,1)**	**(20,0)**
	B	**(0,20)**	**(10,10)**

This may seem like a difficult question but in reality, we just do what we normally do, and a contradiction will emerge. Hence, suppose that player 2 plays A with probability p and B with probability $1 - p$. Then:

$$E[A]_1 = 1p + 20(1 - p)$$

$$E[B]_1 = 0p + 10(1 - p)$$

We set them equal so that player 1 is indifferent between them:

$$1p + 20(1 - p) = 0p + 10(1 - p)$$

$$p + 20 - 20p = 10 - 10p$$

$$p + 10 = 10p$$

$$10 = 9p$$

$$p = \frac{10}{9}$$

However, this is a contradiction because probabilities must be between 0 and 1, and, in this case, $p = \frac{10}{9} > 1$. Therefore, it is impossible to make player 1 indifferent between the two strategies. Hence, no mixed strategy Nash equilibrium is possible here.

3. By making each player indifferent between their not-strictly dominated pure strategies, find the mixed strategy Nash equilibrium of each of the following games.

   a.

		Player 2	
		A	B
Player 1	**A**	**(5,8)**	**(3,2)**
	B	**(2,2)**	**(6,9)**

The standard approach works here. Though things are a bit fiddly since the game is not symmetric. Hence, suppose that player 2 plays A with probability p and B with probability $1 - p$. Then:

$$E[A]_1 = 5p + 3(1 - p)$$

$$E[B]_1 = 2p + 6(1 - p)$$

$$5p + 3(1 - p) = 2p + 6(1 - p)$$

$$5p + 3 - 3p = 2p + 6 - 6p$$

$$2p = 2p + 3 - 6p$$

$$\boldsymbol{p = \frac{1}{2}}$$

Now, suppose that player 1 plays A with probability q and B with probability $1 - q$. Then:

$$E[A]_2 = 8q + 2(1 - q)$$

$$E[B]_2 = 2q + 9(1 - q)$$

$$8q + 2(1 - q) = 2q + 9(1 - q)$$

$$8q + 2 - 2q = 2q + 9 - 9q$$

$$6q = 2q + 7 - 9q$$

$$\boldsymbol{q = \frac{7}{13}}$$

Hence, the Nash equilibrium is $q = \frac{7}{13}, p = \frac{1}{2}$.

b.

		Player 2	
		A	**B**
Player 1	**A**	**(3,2)**	**(6,4)**
	B	**(5,7)**	**(1,1)**

Suppose that player 2 plays A with probability p and B with probability $1 - p$. Then:

$$E[A]_1 = 3p + 6(1 - p)$$

$$E[B]_1 = 5p + (1 - p)$$

$$3p + 6(1 - p) = 5p + (1 - p)$$

$$3p + 6 - 6p = 5p + 1 - p$$

$$-3p + 6 = 4p + 1$$

$$\boldsymbol{p = \frac{5}{7}}$$

Now, suppose that player 1 plays A with probability q and B with probability $1 - q$. Then:

$$E[A]_2 = 2q + 7(1 - q)$$

$$E[B]_2 = 4q + (1 - q)$$

$$2q + 7(1 - q) = 4q + (1 - q)$$

$$2q + 7 - 7q = 4q + 1 - q$$

$$-5q = 3q - 6$$

$$\boldsymbol{q = \frac{3}{4}}$$

Hence, the Nash equilibrium is $q = \frac{3}{4}, p = \frac{5}{7}$.

c.

		Player 2	
		A	**B**
Player 1	**A**	**(2,2)**	**(1,1)**
	B	**(1,1)**	**(2,2)**

Suppose that player 2 plays A with probability p and B with probability $1 - p$. Then:

$$E[A]_1 = 2p + (1 - p)$$

$$E[B]_1 = 1p + 2(1 - p)$$

$$2p + (1 - p) = 1p + 2(1 - p)$$

$$p + 1 = -p + 2$$

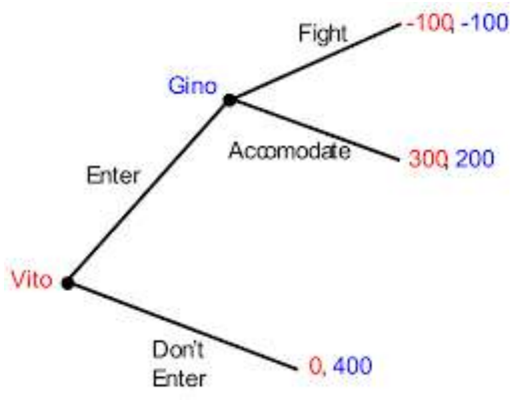
$$2p = 1$$

$$\boldsymbol{p = \frac{1}{2}}$$

We could do the same thing for player 1, but by symmetry, we will get the same result. Hence, the Nash equilibrium is $q = \frac{1}{2}, p = \frac{1}{2}$.

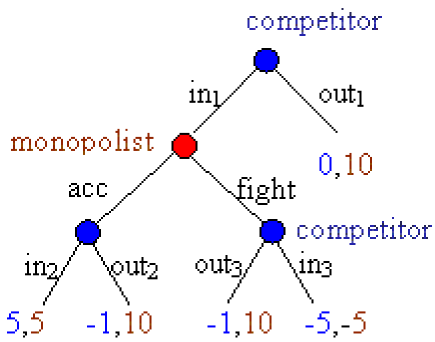
4. Using backward induction, find the sub-game perfect Nash equilibrium of the following sequential games.

a.



We first consider Gino's decision. They can either "Fight" or "Accommodate". If the first, they get a payoff of -100, and if the latter, then the payoff is 200. Hence, Gino would play "Accommodate". Now we consider Vito's decision. If Vito chooses to enter, then by the argument above, Gino would play "Accommodate", thus leading to a payoff of 300 for Vito. If Vito does not enter, they get a payoff of 0. Hence, the sub-game perfect Nash equilibrium is (Enter, Accommodate) with a payoff of (300, 200).

b.



The reasoning here is the same as in the previous game. However, we have three choices being made: first by the competitor, then by the monopolist, and lastly by the competitor again. As before, we start from the end. If the monopolist, accommodates, the competitor will *stay in* since 5 > -1. If the monopolist fights, then the competitor would *leave* since -1 > -5. Now the monopolist decides. Given their knowledge about what the competitor would do, the monopolist chooses to *fight* since 10 > 5. Finally, we come back again to the competitor. They choose so *stay out* since 0 > -1. Therefore, the subgame perfect Nash equilibrium is simply the competitor choosing to stay out of the market leading to a payoff of (0,10).

5. Consider the following normal form game. Identify all the Nash equilibria of this game in both pure and mixed strategies.

		Player 2	
		A	**B**
Player 1	**A**	**(2,6)**	**(2,2)**
	B	**(1,2)**	**(2,4)**

This game looks deceptively simple and can get quite confusing. You can apply the best response approach, but you will run into a problem: player 1 is indifferent between playing A and B, if player 2 opts for B. Don't worry about this. All it means is that either option is a best response. Avoiding this potential pitfall, we can identify two pure strategy Nash equilibria. What about mixed strategy? We again take our usual approach here too.

Suppose that player 2 plays A with probability p and B with probability $1-p$. Then:

$$E[A]_1 = 2p + 2(1-p)$$

$$E[B]_1 = p + 2(1-p)$$

$$2p + 2(1-p) = p + 2(1-p)$$

$$2p + 2 - 2p = p + 2 - 2p$$

$$\boldsymbol{p = 0}$$

Now, suppose that player 1 plays A with probability q and B with probability $1-q$. Then:

$$E[A]_2 = 6q + 2(1-q)$$

$$E[B]_2 = 2q + 4(1-q)$$

$$6q + 2(1-q) = 2q + 4(1-q)$$

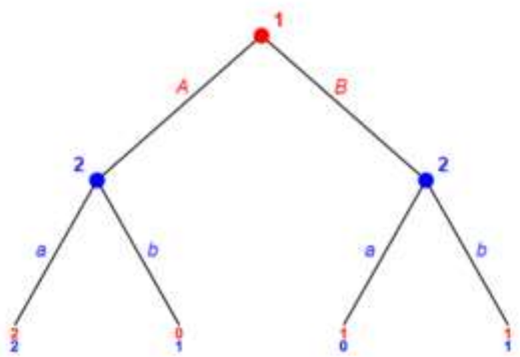
$$6q + 2 - 2q = 2q + 4 - 4q$$

$$4q + 2 = -2q + 4$$

$$\boldsymbol{q = \frac{1}{3}}$$

Hence, the Nash equilibrium is $q = \frac{1}{3}, p = 0$. This is an odd result designed to throw you off! However, it shows that one player may have a pure strategy, while the other has a mixed strategy!

6. Consider the following extended form representations of some sequential games, where player 1 goes first, and player 2 decides afterwards.



   a. Find the sub-game perfect Nash equilibrium.

This quite easy to find. If player 1 plays A, then player 2 will choose to play a. If player 1 plays B, then player 2 will choose to play b. Knowing this, player 1 will choose to play A. Hence the subgame perfect Nash equilibrium is (A, a) yielding a payoff of (2, 2).

b. Transform them into their normal form and find their Nash equilibrium *(Hint: pretend that this game is played simultaneously instead!)*

The transformed simultaneous choice game looks as follows:

		Player 2	
		A	**B**
Player 1	**A**	**(2,2)**	**(0,1)**
	B	**(1,0)**	**(1,1)**

The two Nash equilibria can easily be found using the best response approach.

c. For each of these cases, comment on whether the Nash equilibrium is the same as the sub-game perfect Nash equilibrium.

Notice that in this the game, there are two Nash equilibria, and only one subgame perfect Nash equilibrium. The difference arises because the latter is a refinement of the former. This is because the SPNE respect “sequential rationality” while the former does not!

7. (***Challenge***) Consider the following 3-person simultaneous choice game. Using the best responses of each player, find the pure strategy Nash equilibria of the game.

Player 3 plays A		**Player 2**	
		A	**B**
Player 1	**A**	**(70,70,70)**	**(10,10,23)**
	B	**(60,0,0)**	**(60,65,10)**

Player 3 plays B		**Player 2**	
		A	**B**
Player 1	**A**	**(70,70,60)**	**(10,20,0)**
	B	**(80,50,30)**	**(60,55,5)**

We solve this problem in two stages. First, we find the best responses for players 1 and 2 within each of the two matrices. See the result below.

Player 3 plays A		**Player 2**	
		A	**B**
Player 1	**A**	**(70,70,70)**	**(10,10,23)**
	B	**(60,0,0)**	**(60,65,10)**

Player 3 plays B		**Player 2**	
		A	**B**
Player 1	**A**	**(70,70,60)**	**(10,20,0)**
	B	**(80,50,30)**	**(60,55,5)**

Having found the best responses for players 1 and 2, we can consider the best response of player 3 to the different strategies of player 1 and 2. We thus compare the entry for AA between the first and second table. Repeat this for the other combinations of strategies. I have highlighted 3's best responses in a different colour. The cells where all numbers are highlighted are the Nash equilibria.

Player 3 plays A		**Player 2**	
		A	**B**
Player 1	**A**	**(70,70,70)**	**(10,10,23)**
	B	**(60,0,0)**	**(60,65,10)**

Player 3 plays B		**Player 2**	
		A	**B**
Player 1	**A**	**(70,70,60)**	**(10,20,0)**
	B	**(80,50,30)**	**(60,55,5)**

Hence, there are two Nash equilibria (A, A, A) and (B, B, A).