Game Theory

Preliminary Mathematical Note:

The key concept in game theory is that of the **Nash Equilibrium**. In it, both players are best responding to one another and have not incentive to change or alter their respective strategies. IN the simplest cases, it can be found by looking at a player's **dominant** strategy which they would opt for irrespective of the choices of the other player. For example, in the game below, **A** is a dominant strategy for player 1. You see that regardless of what player 2 does, it is best!

		Player 2	
		A	**B**
Player 1	**A**	**(10,X)**	**(10,X)**
	B	**(5,X)**	**(5,X)**

If both players have a dominant strategy, then a **dominant strategy Nash equilibrium** may emerge. So, for example, in the game below, both players have a dominant strategy of playing **A,** hence leading to a DSNE with a payoff of (10, 10).

		Player 2	
		A	**B**
Player 1	**A**	**(10,10)**	**(10,5)**
	B	**(5,10)**	**(5,5)**

Often, however, neither play has a dominant strategy. In such a case, we find the Nash equilibrium using the **best response approach**. There we consider the best response of each player to each potential strategy of their opponent. In cells where both players are best responding to each other, we have a Nash equilibrium. So, for example, in the game below, the best responses of player 1 are given in green, and the best responses of player 2 are highlighted in yellow. In this case, the Nash equilibrium is again where both players choose **A,** leading to a payoff of (15,10).

		Player 2	
		A	**B**
Player 1	**A**	**(15,10)**	**(10,5)**
	B	**(5,5)**	**(5,10)**

It is sometimes the case that in addition to pure strategies, players can opt for a mixed strategy. To find a mixed strategy Nash equilibrium, you must make the expected payoffs from playing each strategy the same for your opponent. To give an example, consider the game.

		Player 2	
		A	**B**
Player 1	**A**	**(2,4)**	**(-1,-2)**
	B	**(-2,-1)**	**(1,2)**

In this case, there are two pure strategy Nash equilibria which you can find using the best response approach. But what about mixed strategies? To find these we consider the expected payoffs of each player in turn:

Suppose that player 2 plays A with probability p and B with probability $1 - p$. Then, the expected payoff for player 1 of each strategy is:

$$E[A]_1 = 2p - 1(1 - p)$$

$$E[B]_1 = -2p + 1(1 - p)$$

Player 2 wants player 1 to be indifferent between them, hence he sets them equal to one another and solves for p:

$$2p - 1(1 - p) = -2p + 1(1 - p)$$

$$3p - 1 = -3p + 1$$

$$6p = 2$$

$$\boldsymbol{p = \frac{1}{3}}$$

Now, suppose that player 1 plays A with probability q and B with probability $1 - q$. Then, the expected payoff for player 2 of each strategy is:

$$E[A]_2 = 4q - 1(1 - q)$$

$$E[B]_2 = -2q + 2(1 - q)$$

This time, Player 1 wants player 2 to be indifferent between them, hence he sets them equal to one another and solves for q:

$$4q - 1(1 - q) = -2q + 2(1 - q)$$

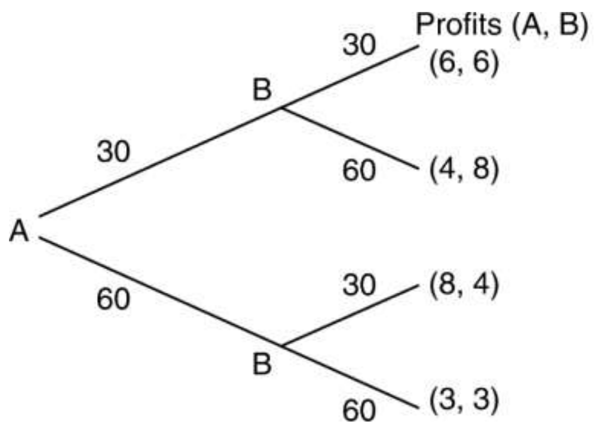
$$5q - 1 = 2 - 4q$$

$$9q = 3$$

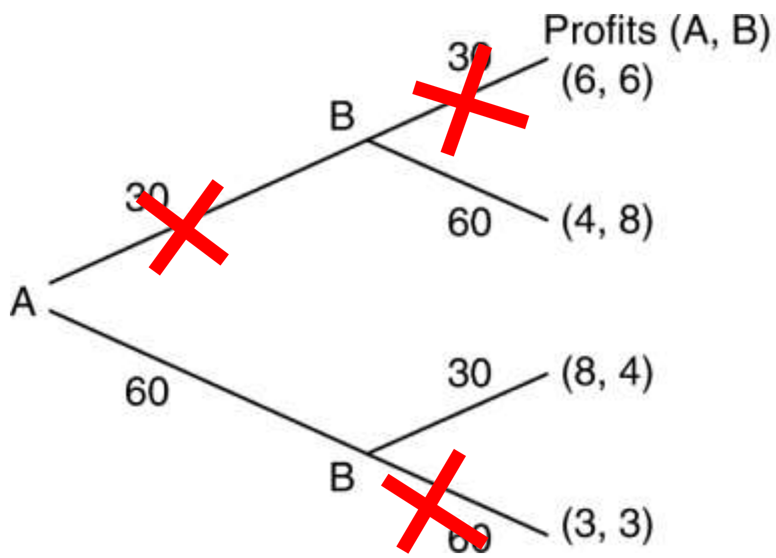
$$\boldsymbol{q = \frac{1}{3}}$$

Hence, the Nash equilibrium is $\boldsymbol{q = \frac{1}{3}, p = \frac{1}{3}}$.

Some games are played sequentially, where one player goes first, and the other chooses second. Such games, such as the one below considered in the tutorials, are solved using **backward induction**.



For this, we look at the decision of the player who chooses last and work backwards by cutting off the irrational branches from the extended form game. Eventually we will be left with a simple decision for the leader to make. This will yield a **subgame perfect Nash equilibrium.** This is a refinement of the notion of a Nash equilibrium which takes the sequential nature of the game into account. In this case, A will play 60, B would respond with 30 leading to a SPNE with payoffs of (8,4), thus illustrating the idea of **first mover advantage!**



Practice Problems

1. Using either the iterative elimination of dominated strategies or by identifying the best responses of each player, find the pure strategy Nash equilibrium of the following games.

a.

		Player 2	
		A	B
Player 1	A	(1,1)	(20,0)
	B	(0,20)	(10,10)

b.

		Player 2	
		A	B
Player 1	A	(3,2)	(6,4)
	B	(5,7)	(1,1)

c.

		Player 2		
		A	B	C
Player 1	A	(5,8)	(3,2)	(8,1)
	B	(2,2)	(6,9)	(7,1)
	C	(1,8)	(2,3)	(4,2)

2. By following the normal procedure when finding a mixed strategy equilibrium, show why in the following prisoners' dilemma game, no such equilibrium is possible:

		Player 2	
		A	B
Player 1	A	(1,1)	(20,0)
	B	(0,20)	(10,10)

3. By making each player indifferent between their not-strictly dominated pure strategies, find the mixed strategy Nash equilibrium of each of the following games.

a.

Player 2

		A	B
Player 1	**A**	**(5,8)**	**(3,2)**
	B	**(2,2)**	**(6,9)**

b.

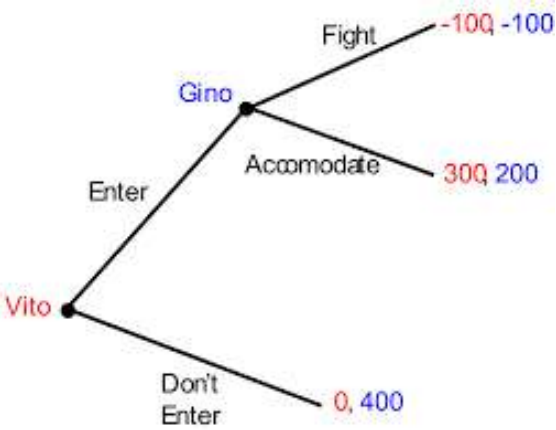
		Player 2	
		A	**B**
Player 1	**A**	**(3,2)**	**(6,4)**
	B	**(5,7)**	**(1,1)**

c.

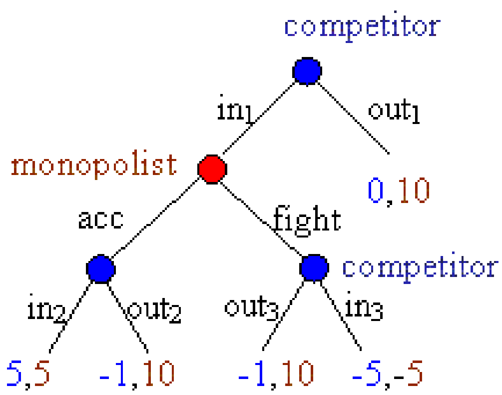
		Player 2	
		A	**B**
Player 1	**A**	**(2,2)**	**(1,1)**
	B	**(1,1)**	**(2,2)**

4. Using backward induction, find the sub-game perfect Nash equilibrium of the following sequential games.

   a.



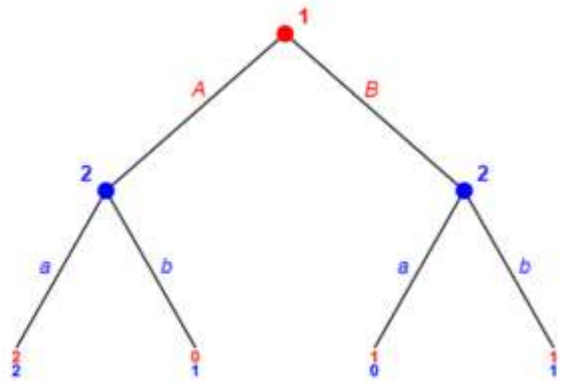
   b.



5. Consider the following normal form game. Identify all the Nash equilibria of this game in both pure and mixed strategies.

		Player 2	
		A	**B**
Player 1	**A**	**(2,6)**	**(2,2)**
	B	**(1,2)**	**(2,4)**

6. Consider the following extended form representations of some sequential games, where player 1 goes first, and player 2 decides afterwards.



   a. Find the sub-game perfect Nash equilibrium.
   b. Transform them into their normal form and find their Nash equilibrium *(Hint: pretend that this game is played simultaneously instead!)*
   c. For each of these cases, comment on whether the Nash equilibrium is the same as the sub-game perfect Nash equilibrium.

7. (***Challenge***) Consider the following 3-person simultaneous choice game. Using the best responses of each player, find the pure strategy Nash equilibria of the game.

Player 3 plays A	**Player 2**

		A	B
Player 1	A	(70,70,70)	(10,10,23)
	B	(60,0,0)	(60,65,10)

Player 3 plays B		Player 2	
		A	B
Player 1	A	(70,70,60)	(10,20,0)
	B	(80,50,30)	(60,55,5)