

## Monopoly and Oligopoly – SOLUTIONS

1. Find the marginal revenue for a monopolist based on the following information:

a.  $p = 250 - Q$

$$TR = (250 - Q)Q \Rightarrow MR = 250 - 2Q$$

b.  $E_d = -2; p = 25$

$$MR = p \left(1 + \frac{1}{\varepsilon}\right) = 10(1 - 0.5) = 5$$

2. Consider the following monopoly markets each with a total cost function of  $C(q) = Q^2 + Q$ . Derive the equilibrium quantity and price for each of these.

a.  $p = 250 - 5Q$

$$MR = MC \Rightarrow 250 - 10Q = 2Q + 1 \Rightarrow Q = 20.75, p = 145.25$$

b.  $10p = 500 - 25Q$

$$MR = MC \Rightarrow 50 - 5Q = 2Q + 1 \Rightarrow Q = 7, p = 32.5$$

c.  $14Q = 490 - 7p$

$$MR = MC \Rightarrow 70 - 4Q = 2Q + 1 \Rightarrow Q = 11.5, p = 47$$

d.  $p = 100 + Q - Q^2$

$$MR = MC \Rightarrow 100 + 2Q - 3Q^2 = 2Q + 1 \Rightarrow Q^2 = 33 \Rightarrow Q = \sqrt{33}, p = 72.74 \text{ (2 d.p.)}$$

e.  $p = 300 - 5Q - Q^2$

$$MR = MC \Rightarrow 300 - 5Q - 3Q^2 = 2Q + 1 \Rightarrow 3Q^2 + 7Q - 299 = 0$$

By quadratic formula to two decimal places:

$$Q = 8.88, p = 230.25$$

3. Suppose that the elasticity of demand is constant at  $E_d = -2$ . Using the appropriate general equation for marginal revenue for a monopoly, explain what happens to price if marginal cost increases by  $\tau$ ? How does this depend on the elasticity of demand?

$$MR = p \left(1 + \frac{1}{-2}\right) \Rightarrow MR = \frac{1}{2}p \Rightarrow p = 2MC$$

Hence suppose that  $MC_2 = MC_1 + \tau$ . Then:

$$\Delta p = p_2 - p_1 = 2MC_2 - 2MC_1 = 2(MC_1 + \tau - MC_1) = 2\tau$$

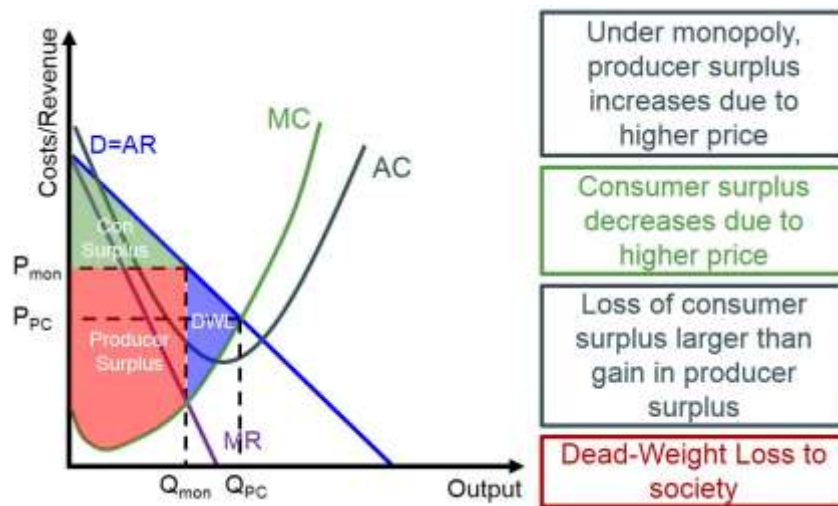
Hence the price increase by twice as much as the increase in marginal cost. More generally:

$$p = \frac{MC}{\left(1 + \frac{1}{\varepsilon}\right)} \Rightarrow \Delta p = \frac{\Delta MC}{\left(1 + \frac{1}{\varepsilon}\right)}$$

Thus, only as  $\varepsilon \rightarrow -\infty$ , does  $\Delta p \rightarrow \Delta MC$ . For all other reasonable values,  $\Delta p > \Delta MC$ .

Note that for this question we are assuming that the price elasticity of demand is a constant, which is possible but often does not hold for many demand functions we normally consider here.

4. Draw a diagram for a typical monopolist with the demand curve (average revenue curve), the MR curve, the MC curve, and the AC curve. Shade and label the consumer and producer surplus, as well as the deadweight loss to society. Label and compare the quantity that would be produced if this were a perfectly competitive market.



5. Consider the following inverse demand function:  $p = 500 - 2Q$ . For this question assume that all firms have a marginal cost of \$2 per unit.
- Compare the incidence of a \$5 specific tax on consumers between a monopoly and a perfectly competitive firm.

Before Tax	After Tax
<p><b>Monopoly:</b></p> $MC = 2$ $MR = 500 - 4Q$ $500 - 4Q = 2$ $Q = 124.5, p = 251$	<p><b>Monopoly:</b></p> $MC = 2 + 5$ $MR = 500 - 4Q$ $500 - 4Q = 7$ $Q = 123.25, p = 253.5$
<p><b>Perfect Competition:</b></p> $p = MC \Rightarrow p = 2$ $2 = 500 - 2Q \Rightarrow Q = 249, p = 2$	<p><b>Perfect Competition:</b></p> $p = MC \Rightarrow p = 2 + 5$ $7 = 500 - 2Q \Rightarrow Q = 246.5, p = 7$

The incidence of taxation under monopoly is then:

$$i_m = \frac{253.5 - 251}{5} = 0.5 = 50\%$$

The incidence of taxation under perfect competition:

$$i_{pc} = \frac{7 - 2}{5} = 1 = 100\%$$

- b. In each of four cases above, how much tax does the government collect?

The total amount of tax is going to be  $\tau Q$ . Hence, we  $\tau_m = 5 \times 123.25 = \mathbf{\$616.25}$ , while  $\tau_{pc} = 5 \times 246.5 = \mathbf{\$1232.50}$

6. Using the inverse demand function  $p = 245 - 5Q$ , and total cost function of  $TC = 2Q^2 + 5$ .
- a. Calculate the equilibrium for a monopoly and a perfectly competitive firm.

*Monopoly:*

$$MR = 245 - 10Q$$

$$MC = 4Q$$

$$245 - 10Q = 4Q \Rightarrow Q = 17.5, p = 157.5$$

*Perfect Competition:*

$$P = MC \Rightarrow p = 4Q$$

$$4Q = 245 - 5Q \Rightarrow Q = \frac{245}{9}, p = \frac{980}{9}$$

- b. What is the deadweight loss associated with monopoly production?

$$DWL = \frac{1}{2}(P - MC)(Q_{pc} - Q_m) = \frac{1}{2} \times (157.5 - (4 \times 17.5)) \times \left(\frac{245}{9} - 17.5\right) \approx \mathbf{425.35}$$

7. Consider the following market inverse demand curve:  $p = 100 - 2Q$ . Assume that all firms in this question, unless stated otherwise, have a total cost function of  $TC = 4Q$ .
- a. Solve for the monopoly equilibrium output and price.

$$MR = 100 - 4Q$$

$$MC = 4$$

$$100 - 4Q = 4 \Rightarrow Q = 24$$

- b. By assuming that  $Q = q_1 + q_2$ , find the Cournot Nash equilibrium.

This example is quite simple because the situation is symmetric:

$$MR_1 = 100 - 4q_1 - 2q_2$$

$$MC = 4$$

$$MC = MR \Rightarrow 100 - 4q_1 - 2q_2 = 4$$

Hence, the reaction functions are:

$$q_1^* = \frac{48 - q_2}{2}, q_2^* = \frac{48 - q_1}{2}$$

Solving these simultaneously:

$$q_1^* = \frac{48 - \frac{48 - q_1}{2}}{2} \Rightarrow q_1^* = q_2^* = \mathbf{16}, p = \mathbf{36}$$

- c. Recall the reaction functions that you found in part (b), are they increasing or decreasing in the quantity produced by the other firm. Explain the intuition behind this.

Notice that  $\frac{\partial q_1^*}{\partial q_2} < 0$ . This implies that the optimal amount of output is decreasing in the production of the other firm. The reason for this is that higher total output reduces the equilibrium price which drives profits towards zero. Hence, it is not in the interest of either firm to produce as much as possible.

- d. Assume instead that quantities are chosen sequentially, find the Nash Stackelberg equilibrium when firm 1 is the leader and firm 2 is the follower.

Since firm 2 is the follower, its reaction function stays the same. However, firm 1 has the first mover advantage and will factor firm 2's plan into their own profit maximisation problem. Hence, we have that:

$$\pi_1 = \left( 100 - 2q_1 - \frac{2(48 - q_1)}{2} \right) q_1 - 4q_1$$

$$\frac{d\pi}{dq_1} = 100 - 4q_1 - 48 + 2q_1 - 4 = 0$$

$$q_1^* = \frac{100 - 48 - 4}{2} = \mathbf{24}, q_2^* = \mathbf{12}, p = \mathbf{28}$$

- e. What difference is difference in terms of market price and profits between these three market structures?

One can work out the profits of each firm. In the Cournot case, both firms have \$512. In the Stackelberg, firm 1 has \$560, while firm 2 only makes 288. Overall, joint profits are lower in the sequential game. Moreover, market price is lower and total output is higher.

- f. Suppose now that firm 1 has a marginal cost of \$8 per unit, while firm 2 has a marginal cost of \$4 per unit.
- i. Find the Cournot Nash equilibrium with these marginal costs.

This is no longer a symmetric scenario, but one where firm 1 now has higher costs relative to the old equilibrium. Note that nothing changes for firm 2. For firm 1, we now have that:

$$MC = MR \Rightarrow 100 - 4q_1 - 2q_2 = 8$$

$$q_1^* = \frac{46 - q_2}{2}$$

Again, solving simultaneously yields:

$$q_1^* = \frac{46 - \frac{48 - q_1}{2}}{2} = \frac{44}{3}, q_2^* = \frac{47}{3}, p = \frac{118}{3}$$

ii. What is the new Stackelberg equilibrium?

$$\pi_1 = \left(100 - 2q_1 - \frac{2(48 - q_1)}{2}\right)q_1 - 8q_1$$

$$\frac{d\pi}{dq_1} = 100 - 4q_1 - 48 + 2q_1 - 8 = 0$$

$$q_1^* = \frac{100 - 48 - 8}{2} = 22, q_2^* = 13, p = 30$$

iii. Comment on why these equilibria differ from those you found in earlier parts of the question.

In the new Cournot equilibrium, firm 1 is now producing less than before. We notice that firm 2 has cutdown production as well, but still produces more than firm 1. In the Stackelberg scenario, a similar event has taken place: firm 1 has reduced its market share. Both cases are explained by the higher marginal costs for firm 1. Since the firm has become less productive, its advantages and foothold in the market has also weakened.

8. Suppose that we have a market with two firms selling a differentiated product. Their respective demand curves are:  $q_1 = 36 - 3p_1 + p_2$  and  $q_2 = 24 + p_1 - 3p_2$ . The marginal cost of both firms is assumed to be \$2.

a. Find the Nash Bertrand equilibrium in prices and quantities.

$$\pi_1 = (36 - 3p_1 + p_2)(p_1 - 2)$$

$$\pi_2 = (24 + p_1 - 3p_2)(p_2 - 2)$$

$$\frac{\partial \pi}{\partial p_1} = (36 - 3p_1 + p_2) - 3(p_1 - 2) = 0$$

$$\frac{\partial \pi}{\partial p_2} = (24 + p_1 - 3p_2) - 3(p_2 - 2) = 0$$

$$(36 - 3p_1 + p_2) - 3(p_1 - 2) = 0 \Rightarrow p_1^* = \frac{42 + p_2}{6}$$

$$(24 + p_1 - 3p_2) - 3(p_2 - 2) = 0 \Rightarrow p_2^* = \frac{30 + p_1}{6}$$

$$p_1^* = \frac{42 + \left(\frac{30 + p_1}{6}\right)}{6} = \frac{282}{35}, q_1 = \frac{636}{35}$$

$$p_2^* = \frac{30 + \frac{282}{35}}{6} = \frac{222}{35}, q_2 = \frac{456}{35}$$

- b. Recall the reaction functions that you found in part (a), are they increasing or decreasing in the price chosen by the other firm. Explain the intuition behind this.

They are increasing as  $\frac{\partial p_1}{\partial p_2} > 0$ . This means that prices are strategic complements.

The reason being that if one firm raises its prices, there is an incentive for the other firm to do the same though by a slightly smaller amount. This is because by raising prices the firms move closer to the monopoly outcome.

- c. Calculate the market share of each firm.

$$s_1 = \frac{\frac{636}{35}}{\frac{636}{35} + \frac{456}{35}} = \frac{53}{91} \approx 58.24\%$$

$$s_2 = \frac{\frac{456}{35}}{\frac{636}{35} + \frac{456}{35}} = \frac{38}{91} \approx 41.76\%$$

- d. What happens to the market share of each firm when firm 2's marginal cost falls to \$1 per unit? What happens to its Lerner index of market power before and after this change?

$$\pi_2 = (24 + p_1 - 3p_2)(p_2 - 1)$$

$$\frac{\partial \pi}{\partial p_2} = (24 + p_1 - 3p_2) - 3(p_2 - 1) = 0$$

$$(24 + p_1 - 3p_2) - 3(p_2 - 1) = 0 \Rightarrow p_2^* = \frac{27 + p_1}{6}$$

$$p_1^* = \frac{42 + \left(\frac{27 + p_1}{6}\right)}{6} = \frac{279}{35}, q_1 = \frac{627}{35}$$

$$p_2^* = \frac{27 + \frac{279}{35}}{6} = \frac{204}{35}, q_2 = \frac{507}{35}$$

Market shares have changed too:

$$s_1 = \frac{\frac{627}{35}}{\frac{627}{35} + \frac{507}{35}} = \frac{209}{378} \approx 55.29\%$$

$$s_2 = \frac{\frac{507}{35}}{\frac{627}{35} + \frac{507}{35}} = \frac{169}{378} \approx 44.71\%$$

The Lerner Index is given by  $\frac{p-MC}{p}$ . For firm 2 this is:

$$L_1 = \frac{\frac{222}{35} - 2}{\frac{222}{35}} \approx 0.68$$

$$L_2 = \frac{\frac{204}{35} - 1}{\frac{204}{35}} \approx 0.83$$

Notice that the Lerner index has increased which suggests that firm 2's market power has risen. Not unexpectedly, the market share of firm 1 has fallen since its relative productivity has declined.

- e. **(Challenge)** Suppose now that prices are chosen sequentially such that firm 1 is the leader and firm 2 is the follower. What is the equilibrium now? Which firm has a greater market share?

This is a sequential variation of the Bertrand equilibrium. The reaction function for firm 2 stays the same. However, the profit function for firm 1 changes.

$$\pi_1 = \left(36 - 3p_1 + \frac{30 + p_1}{6}\right)(p_1 - 2)$$

$$\frac{\partial \pi}{\partial p_1} = \left(36 - 6p_1 + 5 + \frac{p_1}{3}\right) + 6 - \frac{1}{3} = 0$$

$$p_1^* = \frac{\frac{140}{3}}{\frac{17}{3}} = \frac{140}{17}, q_1 = \frac{53}{3}$$

$$p_2^* = \frac{30 + \frac{140}{17}}{6} = \frac{325}{51}, q_2 = \frac{223}{17}$$

$$s_1 = \frac{\frac{53}{3}}{\frac{53}{3} + \frac{223}{17}} = \frac{901}{1570} \approx 57.39\%$$

$$s_2 = \frac{\frac{223}{17}}{\frac{53}{3} + \frac{223}{17}} = \frac{669}{1570} \approx 42.61\%$$

Clearly firm 1 has the advantage. However, notice that the first mover advantage in this Bertrand case is smaller than in the quantity competition scenario.

9. **(Challenge)** Suppose that we observe the following inverse demand function  $p = 36 - 3Q$  such that  $Q = q_1 + q_2 + q_3$ . Assume that all firms have no costs. Solve for the 3-firm Nash

Stackelberg equilibrium where firm 1 chooses its output first, then firm 2 and lastly firm 3. Which firm produces most output? Why?

We start with firm 3:

$$\begin{aligned}\pi_3 &= (36 - 3(q_1 + q_2 + q_3))q_3 \\ \frac{d\pi_3}{dq_3} &= 36 - 3q_1 - 3q_2 - 6q_3 = 0 \\ q_3^* &= \frac{12 - q_1 - q_2}{2}\end{aligned}$$

Next is firm 2:

$$\begin{aligned}\pi_2 &= \left(36 - \left(3q_1 + 3q_2 + \frac{36 - 3q_1 - 3q_2}{2}\right)\right)q_2 \\ \frac{d\pi_2}{dq_2} &= 36 - 3q_1 - 6q_2 + \frac{-36 + 3q_1 + 6q_2}{2} = 0 \\ q_2^* &= \frac{18 - 1.5q_1}{3}\end{aligned}$$

Lastly, we turn to firm 1:

$$\begin{aligned}\pi_1 &= \left(36 - 3\left(q_1 + \frac{18 - 1.5q_1}{3} + \frac{12 - q_1 - \frac{18 - 1.5q_1}{3}}{2}\right)\right)q_1 \\ \pi_1 &= (9 - 0.75q_1)q_1\end{aligned}$$

$$\frac{d\pi_1}{dq_1} = 9 - \frac{3}{2}q_1 = 0 \Rightarrow q_1^* = 6; q_2^* = 3, q_3^* = 1.5, p = 4.5$$

Due to having a first-mover advantage, firm 1 produces the most, then firm 2, while firm 3 produces the least as it makes its output decision last.

10. <sup>1</sup>(Challenge) A differentiated Bertrand duopoly faces the demand curves given by:

$$q_1 = 1 - p_1 + \beta p_2 \text{ and } q_2 = 1 - p_2 + \beta p_1$$

Where  $q_i$  denotes the quantity demanded of each firm,  $p_i$  denotes the chosen price and  $\beta$  is a demand parameter. Assume that the marginal cost of production is zero.

- a. Suppose the firms play only once and set price simultaneously. Derive the best replies for each firm (reaction functions) and determine the Nash equilibrium. How does the equilibrium depend on  $\beta$ ?

$$\begin{aligned}\pi_1 &= (1 - p_1 + \beta p_2)p_1 \Rightarrow \frac{d\pi_1}{dp_1} = 1 - 2p_1 + \beta p_2 = 0 \Rightarrow p_1^* = \frac{1 + \beta p_2}{2} \\ \pi_2 &= (1 - p_2 + \beta p_1)p_2 \Rightarrow \frac{d\pi_2}{dp_2} = 1 - 2p_2 + \beta p_1 = 0 \Rightarrow p_2^* = \frac{1 + \beta p_1}{2} \\ p_1^* &= p_2^* = \frac{1 + \beta \frac{1 + \beta p_1}{2}}{2} = \frac{2 + \beta}{4 - \beta^2}\end{aligned}$$

<sup>1</sup> Question taken from the UCLA Economics, Spring 2005, Second Year PhD Qualifying examination for Industrial Organisation. Good luck, O.B!

As  $\beta \rightarrow \infty$ , both prices tend infinity, this is because demand becomes infinitely elastic with respect to the prices of the other firms' goods.

- b. Determine the most profitable collusive prices that maximises joint profits of both firms.

To maximise joint profits, we simply write the two profit functions together:

$$\Pi = \pi_1 + \pi_2 = (1 - p_1 + \beta p_2)p_1 + (1 - p_2 + \beta p_1)p_2$$

We take both relevant partial derivatives:

$$\frac{\partial \Pi}{\partial p_1} = (1 - 2p_1 + \beta p_2) + \beta p_2 = 0$$

$$\frac{\partial \Pi}{\partial p_2} = (1 - 2p_2 + \beta p_1) + \beta p_1 = 0$$

$$p_1^* = p_2^* = \frac{1}{2 + 2\beta}$$