Costs – SOLUTIONS

Practice Problems

1. Find the *marginal cost* and *average cost* for the following cost functions:

   a. $TC(Q) = Q^2 + 2Q + 1$

   $$\boldsymbol{MC = 2Q + 2; AC = Q + 2 + \frac{1}{Q}}$$

   b. $TC(Q) = 3Q - \frac{1}{Q}$

   $$\boldsymbol{MC = 3 + \frac{1}{Q^2}; AC = 3 - \frac{1}{Q^2}}$$

   c. $TC(Q) = Q^{\frac{1}{2}} + \frac{Q^2+1}{Q-1}$

   $$\boldsymbol{MC = \frac{1}{2}Q^{-\frac{1}{2}} + \frac{Q^2 - 2Q - 1}{(Q-1)}; AC = Q^{-\frac{1}{2}} + \frac{Q^2+1}{Q(Q-1)}}$$

2. Suppose that a firm has two plants with two with the given total cost curves, how much output will be produced at each plant if the firm would like to produce 10 units of a good?

   a. $5q_1^2 - 5$ and $2q_2^2 + 2$

   Equate the marginal cost curves:

   $$10q_1 = 4q_2 \Rightarrow q_1 = 0.4q_2$$

   Substitute into output constraint:

   $$q_1 + q_2 = 10 \Rightarrow 0.4q_2 + q_2 = 1.4q_2 = 10 \Rightarrow \boldsymbol{q_2 = \frac{50}{7}}, q_1 = 10 - \frac{50}{7} = \boldsymbol{\frac{20}{7}}$$

   b. $\frac{3}{2}q_1^3 + 2q_1^2 - 5$ and $2q_2^2 + 1$

   Equate the marginal cost curves:

   $$\frac{9}{2}q_1^2 + 4q_1 = 4q_2$$

Use output constraint and solve simultaneously by substitution:

$$q_2 = 10 - q_1 \Rightarrow \frac{9}{2}q_1^2 + 4q_1 = 4(10 - q_1)$$

Simplifying leads to a quadratic equation:

$$\frac{9}{2}q_1^2 + 8q_1 - 40 = 0$$

Using the quadratic formula (*remembering that there is no negative output*):

$$q_1 = \frac{-8 \pm \sqrt{64 - 4 \times \frac{9}{2} \times -40}}{9} = \frac{-8 \pm 28}{9} = \mathbf{\frac{20}{9}}$$

Hence, our solution is:

$$\boldsymbol{q_1 = \frac{20}{9}, q_2 = \frac{70}{9}}$$

3. Consider the simple production function $Q = KL^{0.5}$, where the wage rate is 5 and the rental rate is 3. Find the short run *average variable cost* and *marginal cost* curves in terms of *output*. Then, find the *long-run average cost* and *marginal cost* again in terms of output. Finally, derive the levels of output for which the short run average variable cost curve is greater than the long-run average cost curve.

Short run:

In the short run capital is fixed. To find the two cost curves we use the identities:

$$AC = \frac{w}{AP_L}, MC = \frac{w}{MP_L}$$

Hence, we have that:

$$AC = \frac{5}{KL^{0.5}/L} = \frac{5\sqrt{L}}{K}$$

By rearranging the production function, we find that:

$$L = \frac{Q^2}{K^2} \Longrightarrow AC = \frac{5\sqrt{L}}{K} \Rightarrow \boldsymbol{AC = \frac{5Q}{K^2}}$$

We follow a similar procedure to find the marginal cost:

$$MC = \frac{5}{0.5KL^{-0.5}} = \frac{10\sqrt{L}}{K} \Rightarrow \boldsymbol{MC = \frac{10Q}{K^2}}$$

Long run:

In the long run, the total cost function is:

$$TC = wL + rK$$

Ideally, we would like it in terms of quantity. However, this means we need to find the optimal ratio of capital to labour. This involves equating the MRTS with the input price ratio:

$$MRTS = \frac{MP_L}{MP_K} = \frac{w}{r} \Rightarrow \frac{0.5KL^{-0.5}}{\sqrt{L}} = \frac{5}{3} \Rightarrow \frac{K}{L} = \frac{10}{3} \Rightarrow K = \frac{10L}{3}$$

We substitute this into the production function:

$$Q = \frac{10L}{3}\sqrt{L} = \frac{10}{3}L^{\frac{3}{2}} \Rightarrow \boldsymbol{L} = (\frac{\mathbf{3}}{\mathbf{10}}\boldsymbol{Q})^{\frac{\mathbf{2}}{\mathbf{3}}}$$

$$Q = K\sqrt{\frac{3}{10}K} = \sqrt{0.3}K^{\frac{3}{2}} \Rightarrow \boldsymbol{K} = (\frac{\mathbf{1}}{\sqrt{\mathbf{0.3}}}\boldsymbol{Q})^{\frac{\mathbf{2}}{\mathbf{3}}}$$

Substituting this into the cost function yields:

$$TC = 5(\frac{3}{10}Q)^{\frac{2}{3}} + 3(\frac{1}{\sqrt{0.3}}Q)^{\frac{2}{3}} = 6.72Q^{\frac{2}{3}}$$

Hence, the average and marginal cost curves are:

$$LRAC = 6.72Q^{-\frac{1}{3}}$$

$$LRMC = 4.48Q^{-\frac{1}{3}}$$

For the last part of the question, we compare the two average cost functions:

$$\frac{3\mathrm{Q}}{\mathrm{K}^2} \geq 6.72\mathrm{Q}^{-\frac{1}{3}}$$

Simplifying yields:

$$Q^{\frac{4}{3}} \geq 2.24K^2$$

$$\boldsymbol{Q} \geq (\mathbf{2.24}\boldsymbol{K}^{\mathbf{2}})^{\frac{\mathbf{3}}{\mathbf{4}}}$$

As a final comment notice that in the long run both cost curves are declining. This is because we are dealing with an increasing returns to scale production function! This is all connected by some deep economic theory!

4. Given the production function $q = KL$ find the long-run expansion path in terms of wage and rent. Use this to find the maximum output that can be produced given input prices and a fixed total cost. Then, again using the expansion path, find the minimum cost that is required to produce a given level of output. Find the minimum cost when output is $q = 100$. Then illustrate the concept of duality by showing that the maximum output that can be produced at that minimum cost is $q = 100$.

To work this out we need to use the condition that MRTS equals the price ratio to find the optimal ratio of inputs:

$$MRTS = \frac{w}{r} \Rightarrow \boldsymbol{\frac{K}{L} = \frac{w}{r}}$$

Hence the expansion path is simply:

$$\boldsymbol{K = \frac{w}{r}L}$$

Substitute into the cost equation:

$$wL + rK = \overline{TC} \Rightarrow wL + wL = \overline{TC} \Rightarrow L = \frac{\overline{TC}}{2w}, K = \frac{\overline{TC}}{2r}$$

Finally, we substitute this into the production function giving us our first result:

$$q = KL = \frac{\overline{TC}}{2w}\frac{\overline{TC}}{2r} \Rightarrow \boldsymbol{q = \frac{\overline{TC}^2}{4wr}}$$

In the other case, we are instead given a target output. Hence, we substitute first into the production function:

$$\bar{q} = KL = \frac{w}{r}L^2 \Rightarrow \boldsymbol{L = \left(\frac{r\bar{q}}{w}\right)^{0.5}}; \boldsymbol{K = \left(\frac{w\bar{q}}{r}\right)^{0.5}}$$

Putting this into the cost function yields:

$$TC = w\boldsymbol{\left(\frac{r\bar{q}}{w}\right)^{0.5} + r\left(\frac{w\bar{q}}{r}\right)^{0.5}}$$

For $q = 100$, the minimum cost is given by:

$$\boldsymbol{TC = 20\sqrt{wr}}$$

If we substitute this into the optimal production function that we found earlier, we get:

$$q = \frac{(20\sqrt{wr})^2}{4wr} = \frac{400wr}{4wr} = \mathbf{100}$$

Hence cost minimisation and output maximisation give us the same solution, thus illustrating the concept of duality nicely! In other words, the maximum output produced using a fixed cost is the same as the minimum cost to produce that same output.

5. Prove that the marginal cost curve intersects the average cost curve at the latter's minimum.

   This proof is the same as the one for the average product and marginal product curves. I do it using product rule, but quotient rule would work just as well:

$$AC = \frac{TC}{q} = TCq^{-1}$$

$$\frac{\partial AC}{\partial q} = \frac{\partial TC}{\partial q}q^{-1} - TCq^{-2} = 0 \Rightarrow \frac{\partial TC}{\partial q} = \frac{TC}{q} \Rightarrow \boldsymbol{MC = AC}$$

6. **(*Challenge*)** Let's combine what we have done above. Suppose a firm has two plants with production functions $q_1 = K_1^{\frac{1}{2}}L_1^{\frac{1}{2}}$ and $q_2 = 2\overline{K}_2^{\frac{1}{2}}L_2^{\frac{1}{2}}$. Suppose further that input prices are fixed at $w = r = 3$, and that the firms needs to produce $q = 100$.
    a. Assume that the firm operates in the short run where $\overline{K} = 16$ for each plant. Find the share of output that is produced at each plant and the total short-run cost necessary to produce this quantity.
    b. Assume now that that we are in the long run. However, we assume that capital stays fixed in plant 2 and becomes variable in plant 1. Find the expansion path of the first plant by equating its MRTS with the input price ratio.
    c. Use this expansion path to find the share of output produced at each plant. Then, compare the total cost of production in the short run with that of the long run. Comment on the difference.

This question is long and hard, but very rewarding! When operating in the short run, we can simplify by substituting in the given amount of capital:

$$q_1 = 4L_1^{\frac{1}{2}} \text{ and } q_2 = 8L_2^{\frac{1}{2}}$$

The question is: how do we get the marginal cost? We use the identity: $MC = \frac{w}{MP_L}$:

$$MC_1 = \frac{3}{2L_1^{-\frac{1}{2}}} = \frac{3}{2}L_1^{\frac{1}{2}}$$

$$MC_2 = \frac{3}{4L_2^{-\frac{1}{2}}} = \frac{3}{4}L_2^{\frac{1}{2}}$$

We ultimately want these equations in terms of output, hence:

$$MC_1 = \frac{3}{8}q_1$$

$$MC_2 = \frac{3}{32}q_2$$

We now do the usual of equating the two marginal costs:

$$MC_1 = MC_2 \Rightarrow \frac{3}{8}q_1 = \frac{3}{32}q_2 \Rightarrow q_1 = \frac{1}{4}q_2$$

We then substitute this into the quantity constraint:

$$q_1 + q_2 = 100 \Rightarrow q_2 + \frac{1}{4}q_2 = 100$$

$$\boldsymbol{q_1} = \mathbf{20\%}, \boldsymbol{q_2} = \mathbf{80\%}$$

The total cost of production can be worked out using the relationship between labour and output:

$$L_1 = \left(\frac{q_1}{4}\right)^2 = 25 \Longrightarrow C_1 = (3 \times 16) + (3 \times 25) = 123$$

$$L_2 = \left(\frac{q_2}{8}\right)^2 = 100 \Longrightarrow C_1 = (3 \times 16) + (3 \times 100) = 348$$

Hence, total short-run cost is given by:

$$TC = C_1 + C_2 = \mathbf{471}$$

We now consider the long run where capital becomes variable. The optimality condition is now:

$$MRTS = \frac{w}{r} = 1$$

Recall that the production functions are now: $q_1 = K_1^{\frac{1}{2}} L_1^{\frac{1}{2}}$ and $q_2 = 8L_2^{\frac{1}{2}}$. This is because the second plant is by assumption keeping capital fixed. We use the trick discussed in the lecture to quickly find the MRTS:

$$\frac{0.5q/_{L_1}}{0.5q/_{K_1}} = 1$$

$$\frac{K_1}{L_1} = 1$$

Hence, the expansion path is:

$$\boldsymbol{K_1} = \boldsymbol{L_1}$$

Having this equality makes everything easier. If you substitute these into the production function, you will see that:

$$q_1 = K_1 = L_1$$

This implies that the total cost function is simply:

$$C_1 = 3q_1 + 3q_1 = 6q_1 \Longrightarrow \boldsymbol{MC_1} = \mathbf{6}$$

We can then equate the two marginal costs again:

$$6 = \frac{3}{32}q_2 \Rightarrow q_2 = 64 \Longrightarrow q_1 = 36$$

Thus, the new percentages are:

$$\boldsymbol{q_1} = \mathbf{36\%}, \boldsymbol{q_2} = \mathbf{64\%}$$

The new total cost function for each plant is:

$$C_1 = 3q_1 + 3q_1 = 3L + 3K = 216$$

$$L_2 = \left(\frac{q_2}{8}\right)^2 = 64 \Longrightarrow C_1 = (3 \times 16) + (3 \times 64) = 240$$

$$TC = C_1 + C_2 = \mathbf{456}$$

We can therefore see that the total cost has fallen from 471 to 456. This is because in the long run, we can adjust capital to its optimal quantity! Notice too that we shifted production away from the plant where capital remained fixed.