Costs

Preliminary Mathematical Note:

When thinking about profit maximisation, a firm must evaluate its costs. The two main types are *fixed costs* and *variable costs*. The former are constant and do not depend on output, while the latter are functions of quantity. For example, for a total cost function of $TC = 10 + 5Q$, the variable cost is equal to $5Q$, while 10 is the fixed cost. We also distinguish between *average* and *marginal cost:*

$$MC = \frac{\partial TC}{\partial q}$$

$$AC = \frac{TC}{q}$$

It is important to note that marginal cost can also be stated as $\Delta TC/_{\Delta Q}$. The difference is that in this case, we are assuming that quantity is discrete. For a firm, we will typically state that costs are given by the isocost line. We note that in the short run, capital is fixed and forms the fixed cost of the firm. Typically, the *isocost line* will be written as:

$$TC = wL + rK \Rightarrow \boldsymbol{K} = -\frac{\boldsymbol{w}}{\boldsymbol{r}}\boldsymbol{L} + \frac{\boldsymbol{TC}}{\boldsymbol{r}}$$

In the discrete case, we can derive an interesting set of results for the short run where capital is fixed:

$$MC = \frac{\Delta TC}{\Delta Q} = \frac{w\Delta L}{\Delta Q}$$

Recall that $MP_L = \frac{\Delta Q}{\Delta L} \Rightarrow \frac{1}{MP_L} = \frac{\Delta L}{\Delta Q}$. We can thus rewrite the equation above as:

$$MC = \frac{w}{MP_L}$$

Recall this from Tutorial/Seminar 4! All these building blocks allow us to think about cost minimisation and output maximisation. We shall explore this using a worked example. Consider the simple case where we have a CRS Cobb-Doughlas production function $q = K^{0.5}L^{0.5}$ with unit input prices. Then, the MRTS and optimal ratio are given by (input price ratio is 1):

$$MRTS = -\frac{MP_L}{MP_K} = -\frac{K}{L} \Longrightarrow K = L$$

This equation $K = L$ is the long-run expansion path. Keeping input prices fixed, the optimal ratio for any level of output will be this one! The specific values for capital labour will depend on the type of problem that we have: are we maximising output subject to an isocost constraints, or are we minimising cost subject to an output constraint?

Suppose it's the first problem which means we know that $K + L = \bar{C}$. We are aiming to produce as much output as possible given a fixed total cost constraint that we must respect. Then, we substitute into this cost function:

$$\boldsymbol{K} = \frac{\overline{\boldsymbol{C}}}{\boldsymbol{2}}; \boldsymbol{L} = \frac{\overline{\boldsymbol{C}}}{\boldsymbol{2}}$$

Substituting this into the production function yields:

$$q = \left(\frac{\bar{C}}{2}\right)^{\frac{1}{2}} \left(\frac{\bar{C}}{2}\right)^{\frac{1}{2}} = \frac{\overline{\boldsymbol{C}}}{\boldsymbol{2}}$$

Hence, we have the maximum output that can be produced using a given total cost. But let's instead suppose that we are given a target level of output $\bar{q}$. Then, instead of substituting into the cost function, we first substitute into the production function:

$$\bar{q} = K^{0.5}K^{0.5} = K = L$$

We then substitute this into the cost function to see that minimum cost required to produce a given level of output:

$$TC = K + L = \boldsymbol{2\bar{q}}$$

Notice how this mirrors what we got previously. This is called *duality*.

Practice Problems

1. Find the *marginal cost* and *average cost* for the following cost functions:

    a. $TC(Q) = Q^2 + 2Q + 1$
    b. $TC(Q) = 3Q - \frac{1}{Q}$
    c. $TC(Q) = Q^{\frac{1}{2}} + \frac{Q^2+1}{Q-1}$

2. Suppose that a firm has two plants with two with the given total cost curves, how much output will be produced at each plant if the firm would like to produce 10 units of a good?

    a. $5q_1^2 - 5$ and $2q_2^2 + 2$
    b. $\frac{3}{2}q_1^3 + 2q_1^2 - 5$ and $2q_2^2 + 1$

3. Consider the simple production function $Q = KL^{0.5}$, where the wage rate is 5 and the rental rate is 3. Find the short run *average variable cost* and *marginal cost* curves in terms of *output*. Then, find the *long-run average cost* and *marginal cost* again in terms of output. Finally, show that for any level of output, the short run average cost curve is greater than or equal to the long-run average cost curve.

4. Given the production function $Q = KL$ find the long-run expansion path in terms of wage and rent. Use this to find the maximum output that can be produced given input prices and a fixed total cost. Then, again using the expansion path, find the minimum cost that is required to produce a given level of output. Find the minimum cost when output is $Q = 100$. Then illustrate the concept of duality by showing that the maximum output that can be produced at that minimum cost is $Q = 100$.

5. Prove that the marginal cost curve intersects the average cost curve at the latter's minimum.

6. **(*Challenge*)** Let's combine what we have done above. Suppose a firm has two plants with production functions $q_1 = K_1^{\frac{1}{2}} L_1^{\frac{1}{2}}$ and $q_2 = 2\overline{K}_2^{\frac{1}{2}} L_2^{\frac{1}{2}}$. Suppose further that input prices are fixed at $w = r = 3$, and that the firms needs to produce $q = 100$.

   a. Assume that the firm operates in the short run where $\overline{K} = 16$ for each plant. Find the share of output that is produced at each plant and the total short-run cost necessary to produce this quantity.
   b. Assume now that that we are in the long run. However, we assume that capital stays fixed in plant 2 and becomes variable in plant 1. Find the expansion path of the first plant by equating its MRTS with the input price ratio.
   c. Use this expansion path to find the share of output produced at each plant. Then, compare the total cost of production in the short run with that of the long run. Comment on the difference.