Production Theory - SOLUTIONS

Practice Problems

1. Find the marginal product of capital, marginal product of labour and marginal rate of technical substitution for the following production functions:
   a. $Q = KL^{0.1}$

$$\boldsymbol{MP_K = L^{0.1};\ MP_L = 0.1KL^{-0.9}; MRTS = -\frac{K}{10L}}$$

   b. $Q = \alpha \ln K + \beta \ln L$

$$\boldsymbol{MP_K = \frac{\alpha}{K};\ MP_L = \frac{\beta}{L}; MRTS = -\frac{\beta K}{\alpha L}}$$

   c. $Q = \frac{KL^2}{K+L^2}$

$$\boldsymbol{MP_K = \frac{L^2}{K+L^2} - \frac{KL^2}{(K+L^2)^2} = \frac{L^4}{(K+L^2)^2};\ MP_L = \frac{2KL}{K+L^2} - \frac{2KL^3}{(K+L^2)^2} = \frac{2K^2L}{(K+L^2)^2}; MRTS = -\frac{2K^2}{L^3}}$$

   d. $Q = K^{\beta} + L$

$$\boldsymbol{MP_K = \beta K^{\beta-1};\ MP_L = 1; MRTS = -\frac{1}{\beta K^{\beta-1}}}$$

2. For each of the following technological changes state whether each one is neutral or non-neutral:
   a. $Q = K^{\frac{1}{2}}L \rightarrow Q = \sqrt{8}K^{\frac{1}{2}}L$

      Neutral: change occurs only to technological constant

   b. $Q = K^{0.1}L^{0.2} \rightarrow Q = K^{0.2}L^{0.4}$

      Neutral: doubling of exponents in this case leads to not change in MRTS (check this!)

   c. $Q = K^{\frac{1}{2}}(5L)^{\frac{2}{7}} \rightarrow Q = 5K^{\frac{1}{2}}L^{\frac{2}{7}}$

      Neutral: change occurs only in the technological constant

   d. $Q = K^{0.1}L^{0.2} \rightarrow Q = 2K^{0.2}L^{0.9}$

Non-neutral: change in exponents is not the same, hence MRTS changes too

3. Consider the production function $Q = 6(KL)^{\frac{2}{3}}$. Show using the *second* partial derivative that this function exhibits diminishing marginal returns to each factor.

$$\frac{\partial Q}{\partial K} = MP_K = 4K^{-\frac{1}{3}}L^{\frac{2}{3}}$$

Notice that this marginal product is decreasing in capital. We can show this more formally using the second partial derivative:

$$\boldsymbol{\frac{\partial^2 Q}{\partial K^2} = -\frac{4}{3}K^{-\frac{4}{3}}L^{\frac{2}{3}} < 0}$$

For all positive input values, this is negative. This implies that the marginal product of capital is decreasing in capital. This means that there is diminishing returns to capital.

4. For the production function $Q = \frac{KL}{K+L}$ find and sketch the average and marginal products of labour. Comment on whether this production function exhibits diminishing marginal returns to labour.

$$AP_L = \frac{Q}{L} = \frac{KL}{(K+L)L} = \boldsymbol{\frac{K}{K+L}}$$

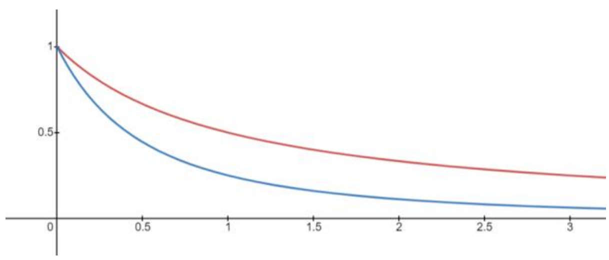
$$MP_L = \frac{K(K+L) - KL}{(K+L)^2} = \boldsymbol{\frac{K^2}{(K+L)^2}}$$

Many ways to discussing diminishing marginal returns. A formal way:

$$\lim_{L \to \infty} \frac{K^2}{(K+L)^2} = 0$$

This step follows simply by noting that $\frac{1}{x} \to 0$, as $x \to \infty$.

The two curves, assuming that $K = 1$, are illustrated as follows:



The top line is the average product curve, while the bottom line is the marginal product curve.

5. Prove that the in the short run, when capital is fixed, the maximum of the average product of labour curve occurs at its intersection with the marginal product of labour.

Start with the definition of average product of labour:

$$AP = \frac{q}{L} = \frac{F(K,L)}{L} = F(K,L)L^{-1}$$

Taking the partial derivative with respect to labour and applying product rule:

$$\frac{\partial AP}{\partial L} = \frac{MP_L}{L} - \frac{F(K,L)}{L^2}$$

Since we are finding the minimum of the average product, we set the first order condition equal to zero:

$$\frac{\partial AP}{\partial L} = \frac{MP_L}{L} - \frac{F(K,L)}{L^2} = 0$$

Solving yields the desired result:

$$\frac{MP_L}{L} - \frac{F(K,L)}{L^2} = 0 \Rightarrow MP_L = \frac{F(K,L)}{L} \Rightarrow \boldsymbol{MP_L = AP_L}$$

6. Consider the Cobb-Doughlas production function $Q = K^{\alpha}L^{\beta}$, where $\alpha, \beta > 0$. For which values of β does this production function exhibit diminishing marginal returns to labour?

Answer: Recall the definition of diminishing marginal returns to labour: the marginal product is decreasing in labour, or the second derivative is negative. Hence, finding the MPL:

$$MP_L = \beta K^{\alpha} L^{\beta-1}$$

One could find the second derivative, but it is sufficient to observe that the marginal product is decreasing in labour only if $\beta - 1 < 0$. Hence the sufficient condition is $\boldsymbol{0 < \beta < 1}$.

7. Consider the production function $Q = K^{\frac{3}{4}}L^{\frac{3}{4}}$. Show that the marginal productivity of each factor is diminishing. Show, however, that for any strictly positive input combination, if the input combination doubles, output more than doubles.

Answer

$$\frac{\partial Q}{\partial K} = \frac{3}{4}K^{-\frac{1}{4}}L^{\frac{3}{4}}; \ \frac{\partial Q}{\partial L} = \frac{3}{4}K^{\frac{3}{4}}L^{-\frac{1}{4}}$$

The two marginal products are symmetric. Both second partial derivatives are negative, and, by inspection, we see that the marginal products diminish with each factor (can be shown via limit too). However, suppose that inputs double then:

$$(2K)^{\frac{3}{4}}(2L)^{\frac{3}{4}} = 2^{\frac{3}{4}}2^{\frac{3}{4}}K^{\frac{3}{4}}L^{\frac{3}{4}} = 2^{\frac{3}{2}}K^{\frac{3}{4}}L^{\frac{3}{4}} > 2K^{\frac{3}{4}}L^{\frac{3}{4}} = 2Q$$

This is because $2^{\frac{3}{2}} > 2$.

8. ***(Challenge)*** Consider the *constant elasticity of substitution* (CES) production function:

$$Q = A[\delta K^{\gamma} + (1-\delta)L^{\gamma}]^{\frac{1}{\gamma}}$$

Where A, γ, δ are constants such that $A > 0, \gamma \neq 0, \gamma < 1$ and $0 < \delta < 1$. Determine the returns to scale of this function. How is your answer modified for the generalised CES production function ($\vartheta > 0$):

$$Q = A[\delta K^{\gamma} + (1-\delta)L^{\gamma}]^{\frac{\vartheta}{\gamma}}$$

Answer: Consider the original production function. Suppose that inputs increase by a factor of k:

$$A[\delta(kK)^{\gamma} + (1-\delta)(kL)^{\gamma}]^{\frac{1}{\gamma}} = A[\delta k^{\gamma}[(K)^{\gamma} + (1-\delta)(L)^{\gamma}]]^{\frac{1}{\gamma}}$$
$$= kA[\delta(kK)^{\gamma} + (1-\delta)(kL)^{\gamma}]^{\frac{1}{\gamma}}$$

Hence, it is constant returns to scale. However, when we consider the generalised CES function, we instead have that:

$$A[\delta(kK)^{\gamma} + (1-\delta)(kL)^{\gamma}]^{\frac{\vartheta}{\gamma}} = k^{\vartheta}A[\delta(kK)^{\gamma} + (1-\delta)(kL)^{\gamma}]^{\frac{\vartheta}{\gamma}}$$

Now returns to scale depend on the parameter ϑ. If $\vartheta = 1$, then it is again constant returns to scale. If $\vartheta > 1$, then increasing returns to scale, while if $\vartheta < 1$ then decreasing.