Production Theory

Preliminary Mathematical Note:

For a production function of the form $Q = F(K, L)$, the *marginal product* of each factor is given by their partial derivatives. Hence:

$$MP_L = \frac{\partial F}{\partial L};\ MP_K = \frac{\partial F}{\partial K}$$

By implicit differentiation or directly by the implicit function theorem, we can show that the *marginal rate of technical substitution* is given by:

$$MRTS = -\frac{MP_L}{MP_K} = -\frac{\partial F/\partial L}{\partial F/\partial K}$$

The average product is given by: $AP_L = \frac{F(K,L)}{L}$. Production functions themselves vary. Some exhibit different types of returns to scale. These can be defined as follows:

- Increasing Returns: $F(\gamma K, \gamma L) > \gamma F(K, L)$
- Constant Returns: $F(\gamma K, \gamma L) = \gamma F(K, L)$
- Diminishing Returns: $(\gamma K, \gamma L) < \gamma F(K, L)$

A firm can undergo a technological change that alters its production function. A *neutral* technological change occurs when the optimal ratio of inputs stays the same from before and after the technological change. This most naturally occurs when the technological parameter increases. For example, when $Q = 5KL \rightarrow Q = 10KL$. Alternatively, and more unusually, the doubling of exponents could also be a neutral change: $Q = \sqrt{xy} \rightarrow Q = xy$. The most thorough check is to see whether the MRTS changes! If so, then the technological change is non-neutral! In our second example above:

$$MRTS_1 = -\frac{0.5x^{-0.5}y^{0.5}}{0.5x^{0.5}y^{-0.5}} = -\frac{y}{x}; MRTS_2 = -\frac{y}{x}$$

Hence, MRTS is the same, and hence the technological change is neutral!

Lastly, in economics, we often assume that production functions exhibit diminishing marginal returns to each input. This means that, when the other input is held fixed, any additional output from increasing the variable unit diminishes. In other words, its marginal product is decreasing and tending towards zero! Think about how this will be reflected in the second partial derivative with respect to that factor!

Practice Problems

1. Find the marginal product of capital, marginal product of labour and marginal rate of technical substitution for the following production functions:
   a. $Q = KL^{0.1}$
   b. $Q = \alpha \ln K + \beta \ln L$
   c. $Q = \frac{KL^2}{K+L^2}$
   d. $Q = K^{\beta} + L$

2. For each of the following technological changes state whether each one is neutral or non-neutral:
   a. $Q = K^{\frac{1}{2}}L \rightarrow Q = \sqrt{8}K^{\frac{1}{2}}L$
   b. $Q = K^{0.1}L^{0.2} \rightarrow Q = K^{0.2}L^{0.4}$
   c. $Q = K^{\frac{1}{2}}(5L)^{\frac{2}{7}} \rightarrow Q = 5K^{\frac{1}{2}}L^{\frac{2}{7}}$
   d. $Q = K^{0.1}L^{0.2} \rightarrow Q = 2K^{0.2}L^{0.9}$

3. Consider the production function $Q = 6(KL)^{\frac{2}{3}}$. Show using the *second* partial derivative that this function exhibits diminishing marginal returns to each factor.

4. For the production function $Q = \frac{KL}{K+L}$ find and sketch the average and marginal products of labour. Comment on whether this production function exhibits diminishing marginal returns to labour.

5. Prove that the in the short run, when capital is fixed, the maximum of the average product of labour curve occurs at its intersection with the marginal product of labour.

6. Consider the Cobb-Doughlas production function $Q = K^{\alpha}L^{\beta}$, where $\alpha, \beta > 0$. For which values of β does this production function exhibit diminishing marginal returns to labour?

7. Consider the production function $Q = K^{\frac{3}{4}}L^{\frac{3}{4}}$. Show that the marginal productivity of each factor is diminishing. Show, however, that for any strictly positive input combination, if the input combination doubles, output more than doubles.

8. ***(Challenge)*** Consider the *constant elasticity of substitution* (CES) production function:

$$Q = A[\delta K^{\gamma} + (1 - \delta)L^{\gamma}]^{\frac{1}{\gamma}}$$

Where A, γ, δ are constants such that $A > 0, \gamma \neq 0, \gamma < 1$ and $0 < \delta < 1$. Determine the returns to scale of this function. How is your answer modified for the generalised CES production function ($\vartheta > 0$):

$$Q = A[\delta K^{\gamma} + (1 - \delta)L^{\gamma}]^{\frac{\vartheta}{\gamma}}$$