

## Consumer Theory – SOLUTIONS

1. For each of following budget constraints find the *marginal rate of transformation*:

a.  $5x + 3y = 58$

$$MRT = -\frac{5}{3}$$

b.  $2x + 7y = 35$

$$MRT = -\frac{2}{7}$$

c.  $4x + y = 24$

$$MRT = -\frac{4}{1} = -4$$

d.  $2x + 8y = 60$

$$MRT = -\frac{2}{8} = -\frac{1}{4}$$

2. For each of the following utility functions, find the *marginal rate of substitution*:

a.  $U(x, y) = \ln x + \ln y^{\frac{1}{2}}$

$$MRS = -\frac{MU_x}{MU_y} = \frac{1/x}{1/2y} = -\frac{2y}{x}$$

b.  $U(x, y) = xy$

$$MRS = -\frac{MU_x}{MU_y} = -\frac{y}{x}$$

c.  $U(x, y) = (x^2 + y^2)^{\frac{1}{2}}$

$$MRS = -\frac{MU_x}{MU_y} = -\frac{x(x^2 + y^2)^{-\frac{1}{2}}}{y(x^2 + y^2)^{-\frac{1}{2}}} = -\frac{x}{y}$$

d.  $U(x, y) = 5x^{\frac{1}{2}}y^{\frac{1}{2}}$

$$MRS = -\frac{MU_x}{MU_y} = -\frac{\frac{5}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}}{\frac{5}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}} = -\frac{y}{x}$$

3. Use your answers to Q1 and Q2 to find the optimal consumption bundle for the following maximisation problems:

a.  $U(x, y) = \ln x + \ln y^{\frac{1}{2}}; 5x + 3y = 58$

$$MRS = MRT \Rightarrow -\frac{2y}{x} = -\frac{5}{3} \Rightarrow 6y = 5x \Rightarrow y = \frac{5}{6}x$$

Substitute into budget constraint:

$$5x + 3\left(\frac{5}{6}x\right) = 58 \Rightarrow \frac{15}{2}x = 58 \Rightarrow x = \frac{116}{15}$$

$$5\left(\frac{116}{15}\right) + 3y = 58 \Rightarrow y = \frac{58}{9}$$

Hence the optimal bundle is  $\left(x = \frac{116}{15}, y = \frac{58}{9}\right)$ .

b.  $U(x, y) = xy; 2x + 7y = 35$

$$MRS = MRT \Rightarrow -\frac{y}{x} = -\frac{2}{7} \Rightarrow 7y = 2x \Rightarrow y = \frac{2}{7}x$$

Substitute into budget constraint:

$$2x + 7\left(\frac{2}{7}x\right) = 35 \Rightarrow 4x = 35 \Rightarrow x = \frac{35}{4}$$

$$2\left(\frac{35}{4}\right) + 7y = 35 \Rightarrow y = \frac{5}{2}$$

Hence the optimal bundle is  $\left(x = \frac{35}{4}, y = \frac{5}{2}\right)$ .

c.  $U(x, y) = (x^2 + y^2)^{\frac{1}{2}}; 4x + y = 24$

$$MRS = MRT \Rightarrow -\frac{x}{y} = -4 \Rightarrow x = 4y$$

Substitute into budget constraint:

$$4(4y) + y = 24 \Rightarrow 17y = 24 \Rightarrow y = \frac{24}{17}$$

$$4x + \left(\frac{24}{17}\right) = 24 \Rightarrow x = \frac{96}{17}$$

Hence the optimal bundle is  $(x = \frac{96}{17}, y = \frac{24}{17})$ .

d.  $U(x, y) = 5x^{\frac{1}{2}}y^{\frac{1}{2}}; 2x + 8y = 60$

$$MRS = MRT \Rightarrow -\frac{y}{x} = -\frac{1}{4} \Rightarrow y = \frac{1}{4}x$$

Substitute into budget constraint:

$$2x + 8\left(\frac{1}{4}x\right) = 60 \Rightarrow 4x = 60 \Rightarrow x = 15$$

$$2(15) + 8y = 60 \Rightarrow y = \frac{15}{4}$$

Hence the optimal bundle is  $(x = 15, y = \frac{15}{4})$ .

4. Suppose that a consumers utility function is given by  $U(x, y) = x + y$ .

a. What is the marginal rate of substitution?

**Answer:** The marginal rate of substitution is 1, because both marginal utilities are equal to 1.

b. What can be said about the type of goods these preferences represent?

**Answer:** The goods are perfect substitutes for this consumer.

c. If prices are  $p_x = 1, p_y = 2$  and  $I = 10$ . How much of each good will the consumer purchase?

**Answer:** These goods are perfect substitutes. Hence the consumer will spend all their income on the relatively cheaper good. In this case,  $p_x < p_y$ . Hence, the consumer will purchase 10 units of good  $x$ .

5. Suppose that a consumers utility function is given by  $U(x, y) = \min \{x, y\}$

a. What can be said about the type of goods these preferences represent?

**Answer:** The goods are perfect complements for this consumer.

b. If prices are  $p_x = 1, p_y = 2$  and  $I = 10$ . How much of each good will the consumer purchase?

**Answer:** The optimal sequence of bundles occurs when  $y = x$ . Hence, by substitution the budget constraint becomes  $x + 2x = 10 \Rightarrow x = y = \frac{10}{3}$

6. Suppose that a consumers utility function is given by  $U(x, y) = 5x + 2y$ .

a. What is the marginal rate of substitution?

**Answer:** In this example, MRS is  $-\frac{MU_x}{MU_y} = -\frac{5}{2} = -2.5$ .

b. What can be said about the type of goods these preferences represent?

**Answer:** These goods are perfect substitutes for this consumer.

c. If prices are  $p_x = 1$ ,  $p_y = 2$  and  $I = 10$ . How much of each good will the consumer purchase?

This case is more complicated than the previous one. However, the logic is still the same as the consumer will only buy the *relatively cheaper good*. How can we figure out which is one relatively cheaper? To do so, compare the value of MRS with the value of the MRT:

$$|MRS| = \frac{5}{2} > \frac{1}{2} = |MRT|$$

This means that the indifference curve is relatively steeper than the budget line. This implies that the relative benefits of the good on the  $x$  – axis is greater than its relative cost. Recall that this is exactly what relative marginal utility and relative price tell us! Hence, the consumer will spend all of their income on good  $x$ , purchasing 10 units of the good. If this is not convincing, see the diagram or try it out by comparing utilities:

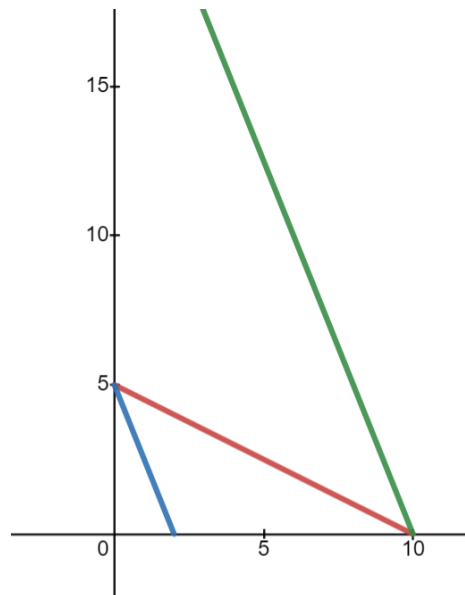
$$U(x, y) = 5 \times 10 + 0 = 50: \text{ if only good } x \text{ is bought}$$

$$U(x, y) = 0 + 2 \times 5 = 10: \text{ if only good } y \text{ is bought}$$

It is clear that only buying good  $x$  is the better choice. However, this may change if the price of this good were to rise sufficiently high. Then,  $MRS < MRT$ , and the opposite would happen as only good  $y$  would be purchased.

d. Use a diagram to support your answer.

**Answer:** See the diagram below.



The red line is the budget constraint. The green line is the indifference curve we are on if we only consume good X, while the blue line is the indifference curve we are on if we only consume good Y. Notice that the green line is further from the origin than the blue line. Hence, only consuming X is the better choice.

7. Suppose that a consumer's utility function is given by  $U(x, y) = \min \{x, 2y\}$ . If prices are  $p_x = 1$ ,  $p_y = 2$  and  $I = 10$ . How much of each good will the consumer purchase?

**Answer:** In this case, we also have perfect complements. However, the ratio is slightly different from the usual case. Here the optimal ratio is  $y = \frac{1}{2}x$ . Substituting this into the budget constraint yields:  $x + 2\left(\frac{1}{2}x\right) = 2x = 10 \Rightarrow x = 5, y = 2.5$ . Here the maximum utility subject to the constraint is 5.

8. **(Challenge)** Suppose that a consumer has the utility function:

$$U(x, y) = -(x - 2)^2 - (y - 2)^2$$

Suppose further that the budget constraint is simply  $x + y = 10$ .

- a. What is the optimal bundle in this case?

**Answer:** The consumer has a *satiation point* at  $x = 2, y = 2$  and utility gets lower and lower the further the consumer moves from this combination of goods.

- b. Will the consumer spend their budget? If not, how much will be left?

**Answer:** No, the consumer will spend their entire budget but will purchase two units of each good. Hence, he has £6 left of his budget.

- c. Suggest a two goods that could be represented by this scenario.

**Answer:** There are many possible suggestions. Since utility falls, we may think that this is some sort of "bad" good. So, for example, cigarettes and alcohol may be good

choices. Ideally, there is a small amount that you may want but utility falls when you go over that amount due to health concerns and other negative effects.

9. **(Challenge)** Suppose that a consumer has the utility function  $U(x, y) = x^{\frac{1}{3}}y^{\frac{2}{3}}$ .

- a. Find the optimal demand for each good  $D_x(p_x, p_y, Y)$  and  $D_y(p_x, p_y, Y)$  in terms of the prices of each good and the total income of the consumer.

The budget constraint can be written in its general form:

$$p_x x + p_y y = Y$$

Hence, the marginal rate of transformation is simply:

$$-\frac{p_x}{p_y}$$

For this Cobb-Douglas utility function, the marginal rate of substitution is:

$$MRS = -\frac{MU_x}{MU_y} = -\frac{\frac{1}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}}}{\frac{2}{3}x^{\frac{1}{3}}y^{-\frac{1}{3}}} = -\frac{y}{2x}$$

We now find the optimal ratio:

$$MRS = MRT \Rightarrow \frac{y}{2x} = \frac{p_x}{p_y} \Rightarrow y = \frac{2p_x x}{p_y}$$

Substituting this into the budget constraint:

$$p_x x + p_y \left( \frac{2p_x x}{p_y} \right) = Y \Rightarrow 3p_x x = Y \Rightarrow x^* = \frac{Y}{3p_x}$$

$$y^* = \frac{2p_x \left( \frac{Y}{3p_x} \right)}{p_y} = \frac{2Y}{3p_y}$$

Hence the two demand functions are:

$$D_x(p_x, p_y, Y) = \frac{Y}{3p_x} \text{ and } D_y(p_x, p_y, Y) = \frac{2Y}{3p_y}$$

- b. Substitute both functions into the original utility function to find the *indirect utility function*  $U^*(p_x, p_y, Y)$ .

Recall that the utility function is  $U(x, y) = x^{\frac{1}{3}}y^{\frac{2}{3}}$ . We substitute our demand equations into it to find the indirect utility function:

$$U^*(p_x, p_y, Y) = \left( \frac{Y}{3p_x} \right)^{\frac{1}{3}} \left( \frac{2Y}{3p_y} \right)^{\frac{2}{3}}$$

- c. Roy's identity states that one can recover the demand functions for each good using the indirect utility function using the formulas:

$$D_x(p_x, p_y, Y) = -\frac{\frac{\partial U^*}{\partial p_x}}{\frac{\partial U^*}{\partial Y}} \text{ and } D_y(p_x, p_y, Y) = -\frac{\frac{\partial U^*}{\partial p_y}}{\frac{\partial U^*}{\partial Y}}$$

Verify that these are indeed correct.

Start with the indirect utility function:

$$U^*(p_x, p_y, Y) = \left(\frac{Y}{3p_x}\right)^{\frac{1}{3}} \left(\frac{2Y}{3p_y}\right)^{\frac{2}{3}} = Y \left(\frac{1}{3p_x}\right)^{\frac{1}{3}} \left(\frac{2}{3p_y}\right)^{\frac{2}{3}}$$

Let's find each of the necessary partial derivatives:

$$\frac{\partial U^*}{\partial Y} = \left(\frac{1}{3p_x}\right)^{\frac{1}{3}} \left(\frac{2}{3p_y}\right)^{\frac{2}{3}}$$

$$\frac{\partial U^*}{\partial p_x} = -\frac{1}{3} 3^{-\frac{1}{3}} Y p_x^{-\frac{4}{3}} \left(\frac{2}{3p_y}\right)^{\frac{2}{3}}$$

$$\frac{\partial U^*}{\partial p_y} = -\frac{2}{3} \left(\frac{2}{3}\right)^{\frac{2}{3}} Y p_y^{-\frac{5}{3}} \left(\frac{1}{3p_x}\right)^{\frac{1}{3}}$$

For good  $x$ :

$$\begin{aligned} D_x(p_x, p_y, Y) &= \frac{\frac{1}{3} 3^{-\frac{1}{3}} Y p_x^{-\frac{4}{3}} \left(\frac{2}{3p_y}\right)^{\frac{2}{3}}}{\left(\frac{1}{3p_x}\right)^{\frac{1}{3}} \left(\frac{2}{3p_y}\right)^{\frac{2}{3}}} = \frac{\frac{1}{3} 3^{-\frac{1}{3}} Y p_x^{-\frac{4}{3}}}{\left(\frac{1}{3p_x}\right)^{\frac{1}{3}}} = \frac{\frac{1}{3} 3^{-\frac{1}{3}} Y p_x^{-\frac{4}{3}}}{3^{-\frac{1}{3}} p_x^{-\frac{1}{3}}} = \frac{1}{3} 3^{-\frac{1}{3}} Y p_x^{-\frac{4}{3}} 3^{\frac{1}{3}} p_x^{\frac{1}{3}} \\ &= \frac{1}{3} Y p_x^{-1} = \frac{Y}{3p_x} \end{aligned}$$

Hence, Roy's identity is verified for good  $x$  as required.

For good  $y$ :

$$\begin{aligned} D_y(p_x, p_y, Y) &= \frac{\frac{2}{3} \left(\frac{2}{3}\right)^{\frac{2}{3}} Y p_y^{-\frac{5}{3}} \left(\frac{1}{3p_x}\right)^{\frac{1}{3}}}{\left(\frac{1}{3p_x}\right)^{\frac{1}{3}} \left(\frac{2}{3p_y}\right)^{\frac{2}{3}}} = \frac{\frac{2}{3} \left(\frac{2}{3}\right)^{\frac{2}{3}} Y p_y^{-\frac{5}{3}}}{\left(\frac{2}{3p_y}\right)^{\frac{2}{3}}} = \frac{2}{3} \left(\frac{2}{3}\right)^{\frac{2}{3}} Y p_y^{-\frac{5}{3}} \left(\frac{2}{3p_y}\right)^{-\frac{2}{3}} \\ &= \frac{2}{3} Y p_y^{-\frac{5}{3}} (p_y)^{\frac{2}{3}} = \frac{2}{3} Y p_y^{-1} = \frac{2Y}{3p_y} \end{aligned}$$

Hence, Roy's identity is verified for good  $y$  as required.