Consumer Theory

Preliminary Mathematical Note:

Remember that the slope of the budget line, called the *marginal rate of transformation*, is given by:

$$-\frac{p_x}{p_y}$$

The slope of the indifference curves is given by the *marginal rate of substitution*:

$$-\frac{MU_x}{MU_y}$$

When maximising utility, we will often want to set $MRT = MRS$, and use the budget constraint to for the optimal combination (bundle) of goods.

Let's go through a step-by-step example:

$$U(x, y) = x^{\frac{1}{4}}y^{\frac{3}{4}} \text{ subject to } 3x + 2y = 9$$

In this the budget constraint is already written out. We are told that $p_x = 3, p_y = 2$ and income is $Y = 9$. We proceed in 4 steps.

Step 1 – Find the marginal rate of transformation:

$$2y = 9 - 3x \Rightarrow y = \frac{9}{2} - \frac{3}{2}x$$

The MRT is the slope of the budget constraint. Hence, $MRT = -\frac{3}{2}$.

Step 2 – Find the marginal rate of substitution:

$$(x, y) = x^{\frac{1}{4}}y^{\frac{3}{4}}$$

We need to find the marginal utility of each good. To do so we partially differentiate the utility function with respect to each good.

$$MU_x = \frac{1}{4}x^{-\frac{3}{4}}y^{\frac{3}{4}}$$

$$MU_y = \frac{3}{4}x^{\frac{1}{4}}y^{-\frac{1}{4}}$$

Recall that the MRS is just the negative ratio of these two equations:

$$MRS = -\frac{\frac{1}{4}x^{-\frac{3}{4}}y^{\frac{3}{4}}}{\frac{3}{4}x^{\frac{1}{4}}y^{-\frac{1}{4}}} = -\frac{x^{-1}y}{3} = -\frac{y}{3x}$$

Step 3 – MRT = MRS

Next, we find the place where the MRS is equal to MRT. This will give us the optimal ratio of goods in the optimal bundle:

$$-\frac{y}{3x} = -\frac{3}{2} \Rightarrow 2y = 9x \Rightarrow y = \frac{9}{2}x$$

Step 4 – Substitute into budget constraint to find optimum

$$3x + 2\left(\frac{9}{2}x\right) = 9 \Rightarrow 3x + 9x = 9 \Rightarrow 12x = 9 \Rightarrow x = \frac{9}{12} = \frac{3}{4}$$

We now substitute the optimal value for x to find y:

$$3\left(\frac{3}{4}\right) + 2y = 9 \Rightarrow 9 + 8y = 36 \Rightarrow 8y = 27 \Rightarrow y = \frac{27}{8}$$

Hence, the optimal bundle is $\left(\boldsymbol{x = \frac{3}{4}, y = \frac{27}{8}}\right)$

Practice Problems

1. For each of following budget constraints find the *marginal rate of transformation*:

   a. $5x + 3y = 58$
   b. $2x + 7y = 35$
   c. $4x + y = 24$
   d. $2x + 8y = 60$

2. For each of the following utility functions, find the *marginal rate of substitution*:
   a. $U(x,y) = \ln x + \ln y^{\frac{1}{2}}$
   b. $U(x,y) = xy$
   c. $U(x,y) = (x^2 + y^2)^{\frac{1}{2}}$
   d. $U(x,y) = 5x^{\frac{1}{2}}y^{\frac{1}{2}}$

3. Use your answers to Q1 and Q2 to find the optimal consumption bundle for the following maximisation problems:
   a. $U(x,y) = \ln x + \ln y^{\frac{1}{2}}$; $5x + 3y = 58$
   b. $U(x,y) = xy; 2x + 7y = 35$
   c. $U(x,y) = (x^2 + y^2)^{\frac{1}{2}}$; $4x + y = 24$
   d. $U(x,y) = 5x^{\frac{1}{2}}y^{\frac{1}{2}}; 2x + 8y = 60$

4. Suppose that a consumers utility function is given by $U(x,y) = x + y$.
   a. What is the marginal rate of substitution?
   b. What can be said about the type of goods these preferences represent?
   c. If prices are $p_x = 1, p_y = 2$ and $I = 10$. How much of each good will the consumer purchase?

5. Suppose that a consumers utility function is given by $U(x,y) = \min\ \{x,y\}$
   a. What can be said about the type of goods these preferences represent?

b. If prices are $p_x = 1, p_y = 2$ and $I = 10$. How much of each good will the consumer purchase?

6. Suppose that a consumers utility function is given by $U(x, y) = 5x + 2y$.
   a. What is the marginal rate of substitution?
   b. What can be said about the type of goods these preferences represent?
   c. If prices are $p_x = 1, p_y = 2$ and $I = 10$. How much of each good will the consumer purchase?
   d. Use a diagram to support your answer.

7. Suppose that a consumers utility function is given by $U(x, y) = \min\{x, 2y\}$. If prices are $p_x = 1, p_y = 2$ and $I = 10$. How much of each good will the consumer purchase?

8. ***(Challenge)*** Suppose that a consumer has the utility function:
$$U(x, y) = -(x-2)^2 - (y-2)^2$$
Suppose further that the budget constraint is simply $x + y = 10$.
   a. What is the optimal bundle in this case?
   b. Will the consumer spend their budget? If not, how much will be left?
   c. Suggest a two goods that could be represented by this scenario.

9. (***Challenge***) Suppose that a consumer has the utility function $U(x, y) = x^{\frac{1}{3}}y^{\frac{2}{3}}$.
   a. Find the optimal demand for each good $D_x(p_x, p_y, Y)$ and $D_y(p_x, p_y, Y)$ in terms of the prices of each good and the total income of the consumer.
   b. Substitute both functions into the original utility function to find the *indirect utility function* $U^*(p_x, p_y, Y)$.
   c. Roy's identity states that one can recover the demand functions for each good using the indirect utility function using the formulas:
$$D_x(p_x, p_y, Y) = -\frac{\frac{\partial U^*}{\partial p_x}}{\frac{\partial U^*}{\partial Y}} \text{ and } D_x(p_x, p_y, Y) = -\frac{\frac{\partial U^*}{\partial p_y}}{\frac{\partial U^*}{\partial Y}}$$
   Verify that these are indeed correct.