Mathematics for Economists

These short notes are primarily a tool for revision and cannot replace a textbook or formal course in mathematics. The exercises can be used by all students. For partial differentiation, there are additional challenge exercises for more advanced students. If you have any questions, comments, or suggestions, please do not hesitate to email me: grigory.aleksin@rhul.ac.uk

For a textbook treatment of these topics and more questions see:

1. Mathematics for Economists, Pemberton and Rau
2. Mathematics for Economists, Simon and Blume
3. Introduction to Mathematical Economics, Dowling (Schaum's Outlines)

Equations and Algebra

In economics, we often work with equations and need to be able to manipulate them quickly and effectively. The most basic of these is the linear equation that takes the form:

$$y = mx + c$$

Here, m is the constant slope of the line and c is the y-intercept. These can be extended to quadratic equations which have a characteristic parabola shape. Quadratic equations have two forms. For example:

$$(x+5)(x-3) = x^2 + 2x = 15$$

The "factorised" form is particularly useful when solving for the roots of the equation, that is those place where it crosses the y-axis. More formally, for a given quadratic equation $ax^2 + bx + c$, the solution is given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The proof of this result comes from completing the square on the quadratic equation:

$$ax^2 + bx + c = x^2 + \frac{b}{a}x + \frac{c}{a} = (x + \frac{b}{2a})^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$(x + \frac{b}{2a})^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

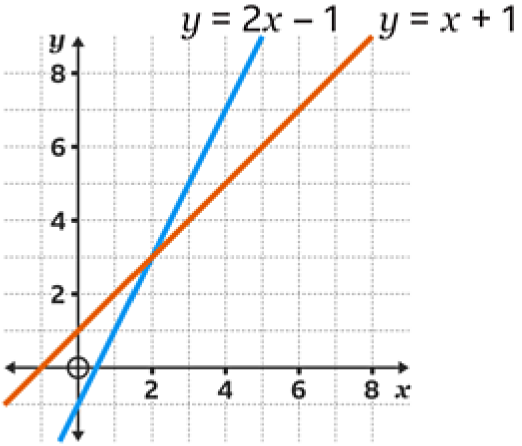
$$x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

We have nearly gone there, just some simplification required:

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula will provide you with both real roots of the quadratic if they exist.

Sometimes, instead of one, we may have several equations. In the simplest case, these represent lines that can be graphed. Particularly, when studying supply and demand, we may need to find the point at which the lines intersect. This means we have to solve the equations “simultaneously”!



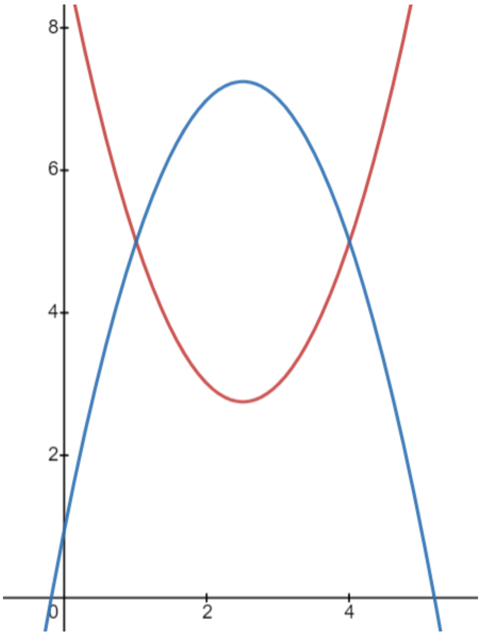
For example, suppose we have two lines $y = 2x - 1$ and $y = x + 1$. At the point of their intersection both the y and x coordinates are the same. Hence, we can make the lines equal:

$$2x - 1 = x + 1 \Rightarrow x = 2$$

Having found the x-coordinate of the intersection, we can substitute 2 in either equation to find the respective y-coordinate:

$$(2 \times 2) - 1 = 2 + 1 = 3$$

Hence the point of intersection is $(2, 3)$. This example was pretty easy. When doing more advanced mathematics, this exercise can be made much more complicated. Suppose now we have two quadratic equations (pictured below): $y = x^2 - 5x + 9$ and $y = -x^2 + 5x + 1$.



This time we have two intersections! However, the process is the same! The y-coordinates at the intersections for both curves coincide. Hence, we can just equate the two curves as before:

$$-x^2 + 5x + 1 = x^2 - 5x + 9 \Rightarrow 2x^2 - 10x + 8 = 0 \Longrightarrow (2x - 2)(x - 4) = 0$$

Hence, the two coordinates of the intersection are simply $x = 1$ and $x = 4$. Plugging these into one of the quadratic equations gives us:

$$1 - 5 + 9 = 5 \Rightarrow (1, 5); 16 - 20 + 9 = 5 \Rightarrow (4, 5)$$

Hence, the intersections occur at two points: $(1, 5)$ and $(4, 5)$.

Apart from quadratics, we may have equations with differing “powers”. For example, $y = x^3$ or $y = x^{\frac{1}{2}}$ or even $x^{-\frac{25}{7}}$. In all these case, x has a different power or index. There are a few rules that can be used to manipulate indices. These are:

- $x^a x^b = x^{a+b}$
    - $x^{\frac{1}{2}} x^{\frac{1}{4}} = x^{\frac{3}{4}}$
- $\frac{x^a}{x^b} = x^{a-b}$
    - $\frac{x^{\frac{1}{3}}}{x^{-\frac{2}{3}}} = x^{\frac{1}{3} - -\frac{2}{3}} = x^{\frac{1}{3} + \frac{2}{3}} = x$
- $(x^a)^b = x^{ab}$
    - $(x^2)^{\frac{1}{4}} = x^{\frac{1}{2}} = \sqrt{x}$

It is important to note that we need to solve simultaneous equations with these indices too!

Practice Problems

1. Factorise and solve the following quadratic equations:
    a. $x^2 + 5x + 6$
    b. $x^2 - 4$
    c. $x^2 + 6x + 9$
    d. $5x^2 + 13x + 6$
2. Solve the following linear simultaneous equations:
    a. $y = 5x + 6; y = -3x + 2$
    b. $2y = 4x + 2; -\frac{1}{2}y = \frac{5}{2}x - 3$
    c. $3x + 2y = z + 11; 2x + z = 7 + 3y; 5x + y - 2x = 12$
3. Solve the following quadratic simultaneous equations:
    a. $y = x + 3; y = x^2 + 5x - 2$
    b. $y = x^2 + 5; y = -x^2 + 15$
    c. $x^2 + y^2 = 25; y = x + 5$
4. Simplify the following expressions:
    a. $\frac{x^{-\frac{1}{3}}}{x^{-1}}$
    b. $x^2 x^{-\frac{1}{7}} x^{\frac{1}{2}}$
    c. $(2^x)^2 4^{x^{\frac{1}{2}} x^{\frac{1}{3}}} 2^{x^3}$
5. (***Challenge***) Solve the following equations simultaneously:

$$y = x^{\frac{1}{2}} - 3; 5y = -x^{\frac{1}{2}} - 5$$

(***Challenge***) When using the Lagrangian Multiplier Method (*do not worry about what this is*), we often end up with a collection of equations that need to be solved simultaneously. Find *all* the solutions to the following (ignoring λ):

$$2x = \lambda(10x + 6y)$$
$$2y = \lambda(10y + 6x)$$
$$5x^2 + 6xy + 5y^2 = 1$$

6. (***Challenge***) Solve the following simultaneous equations:

$$\frac{8^x}{128} = \frac{1}{4^y};\ \frac{49^x}{7^y} = 1$$

Functions and Inverses

A function is a rule that assigns an input or multiple inputs a unique output. For example, we may often write $y = f(x)$. This is a simple way of writing that there is some relationship between an independent variable x and a dependent variable y. In mathematics, we often write theorems and results without explicitly stating the rule that we are working with. This is because many theorems hold generally. Hence, we need a way to write down these relationships in an abstract fashion. Of course, as we shall see later, we do not need to restrict ourselves to functions of one variable!

For example, a useful function that we often encounter in mathematical analysis is the *absolute value* function denoted by $y = |x|$. It converts negatively valued to their positive counterparts. Hence, $|-5| = 5$. It is used to analyse inequalities: $|x - 5| < \varepsilon \Leftrightarrow -\varepsilon < x - 5 < \varepsilon \Leftrightarrow 5 - \varepsilon < x < 5 + \varepsilon$. Because of this, it helps us understand the distance between any two points.

Sometimes, we can combine functions together. These are called *composite functions*. For example, suppose that $f(x) = 5x + 1$ and $g(x) = x^2$, then:

$$f(g(x)) = 5(x^2) + 1 \text{ and } g(f(x)) = (5x + 1)^2$$

What's going on here? In the first case, we apply the rules in the following order: $x \to g(x) \to f(x)$, while in the second case we do: $x \to f(x) \to g(x)$.

Functions can have many different properties. In economics, we often encounter *homogeneous functions*! A homogeneous function of degree k is such that:

$$F(\alpha x, \alpha y) = \alpha^k F(x, y)$$

This may seem a little abstract. An example may make things clearer. Consider the following Cobb-Doughlas production function:

$$F(x, y) = x^{\frac{1}{2}}y^{\frac{1}{2}}$$

Suppose we multiply each input by $\alpha > 0$:

$$F(\alpha x, \alpha y) = (\alpha x)^{\frac{1}{2}}(\alpha y)^{\frac{1}{2}} = \alpha^{\frac{1}{2}}\alpha^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{1}{2}} = \alpha x^{\frac{1}{2}}y^{\frac{1}{2}} = \alpha F(x, y)$$

Hence, the Cobb-Doughlas production function with constant returns to scale is homogeneous of degree 1! Let's see what happens if we give it increasing returns to scale:

$$F(x, y) = xy$$

Again, suppose we multiply each input by $\alpha > 0$:

$$F(\alpha x, \alpha y) = (\alpha x)(\alpha y) = \alpha^2 xy = \alpha^2 F(x, y)$$

This time we see that the function is homogeneous of degree 2! Interestingly, this reveals quite a deep result. Namely, that a homogeneous function of degree 1 is associated with constant returns to scale.

Another property of function is *monotonicity*. Put simply, a monotonic function always goes up or always goes down – it moves in one direction. For example, the function $f(x) = x$ is monotonically increasing, while $f(x) = x^2$ is non-monotonic. Monotonicity is a useful property because it implies that a function has an *inverse.* An inverse of a function is a rule that assigns an input to the output of a function. For example, for the function $f(x) = 5x$, the inverse function is $f^{-1}(x) = \frac{x}{5}$. The key point about inverse functions is that we can combine with the original function to give the following identity:

$$f^{-1}\big(f(x)\big) = x$$

In our example: $f^{-1}\big(f(x)\big) = \frac{(5x)}{5} = x$, as required.

Practice Problems

1. Solve the following absolute value equation: $|2x - 4| = 10$
2. Given the functions $f(x) = x^3 + 5x + 1$ and $g(x) = x^{\frac{1}{3}} + 2x^3$ state $f(g(x))$ and $g(f(x))$ as simply as possible.
3. State the degree of the following homogeneous functions:
    a. $F(x, y,) = x^{\frac{1}{3}} y^{\frac{2}{3}}$
    b. $F(x, y) = \min\,\{x, y\}$
    c. $F(x, y) = x^2 y^2$
    d. $F(x, y) = x^3 + y^3$
4. Are the following functions monotonous? (*Hint:* Use *Desmos* to graph each one if unsure)
    a. $y = x^3$
    b. $y = x^2 + 3$
    c. $y = 5x - 8$
    d. $y = x^{\frac{1}{2}}$
    e. $y = e^x$
5. Find the inverse function of the following functions and verify your solutions using the inverse function identity:
    a. $f(x) = 5x + 7$
    b. $f(x) = -x + 3$
    c. $f(x) = x^{\frac{1}{3}}$
6. Does the function $f(x) = x^2$ have an inverse function? Why, or why not?

Limits

In pure mathematics, the concept of a limit is fundamental to understanding the continuity and differentiability properties of functions. We will not go fully into those details here but provide the rudimentary information needed for the “Introduction to Microeconomics” course. The idea of a limit is very simple: we are just looking at where a function or a sequence is tending to. For example, consider the sequence:

$$\frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

We can see that this sequence is getting smaller and smaller and smaller. Notice that it will never reach zero yet the further down we go, the closer and closer the sequence moves towards it. Due to this tendency, we say that the ***limit*** of the sequence is zero. In mathematical notation:

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

What this says is that if we make n get bigger and bigger, the sequence will get closer and closer to zero. Importantly, the sequence will *keep* getting closer and stay there. We can say this even more formally as: for any small positive number $\varepsilon > 0$ that we choose, there will always be a number N such that for all members of the sequence $n > N$, it will be the case that $-\varepsilon < \frac{1}{n} - 0 < \varepsilon$.

We have a similar idea for functions. Suppose that $f(x) = \frac{1}{x}$. Here we have something similar to the example above. Suppose that $x \to \infty$, that is we let x get bigger and bigger, we can see that the value of the function decreases and again tends to zero. Hence, we write:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x} = 0$$

In many cases, we can use a “plug-in” approach. We plug in the value to which we are letting x tend and see what happens. For most applications in economics, limits are quite easy to find.

Practice Problems

Evaluate the following limits:

1. $\lim_{x \to \infty} \frac{1}{x^2}$
2. $\lim_{x \to 2} (x^2 - 4)$
3. $\lim_{x \to 2} \frac{x+1}{x-1}$
4. $\lim_{x \to 3} \frac{x-3}{x^2-5x+6}$ (*Hint:* try factorisation)
5. $\lim_{x \to 2} \frac{x^2-4}{x-2}$ (*Hint:* try factorisation)

Logarithms and Exponentials

Logarithms are an important tool for economists as they help us simplify difficult expression, transform econometric models, and study growth rates. A logarithm is defined as follows:

$$\log_a y = x \Leftrightarrow y = a^x$$

In words, a logarithm tells us the value of the exponent needed so that a^x yields a specific value y. The constant a is called the base of the logarithm. It varies and the one we choose often depends on the circumstances. That being said, a common choice is the number e. Hence, we have that:

$$\log_e y = x \Leftrightarrow y = e^x$$

When working with this type of logarithm, we make use of the notation $\ln y$ for simplicity, and we call it the *natural logarithm*. More generally, logarithms have several useful properties:

- $\ln\, xy = \ln x + \ln y$
- $\ln \frac{x}{y} = \ln x - \ln y$
- $\ln x^n = n \ln x$

Why are logarithms useful? They help us simplify complicated expressions. For example, consider the production function $Q = AK^{\frac{1}{2}}L^{\frac{1}{2}}$. If we take natural logarithms of both sides, we see that:

$$\ln Q = \ln\left(AK^{\frac{1}{2}}L^{\frac{1}{2}}\right) = \ln A + \frac{1}{2}\ln K + \frac{1}{2}\ln L$$

Notice how the equation now looks linear! This is much easier to work with. The natural logarithm brings us to a useful function called the *exponential*. It is given by $y = e^x$. As you study derivatives, you will see that this function has a cool feature, namely that: $y = \frac{dy}{dx} = e^x$! The function is its own gradient!

Practice Problems

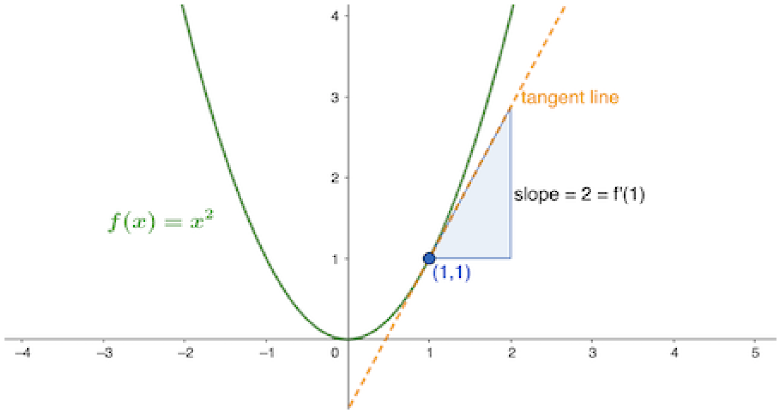
1. Simplify the following logarithmic expressions:
    a. $\ln e^x$
    b. $\ln(x^2 y)$
    c. $\ln\left(\frac{x^{-2}}{y}\right)$
    d. $\ln(x^2 e^{-x})$

2. It can be shown that an equation that is in log-log form is such that the coefficients are elasticities. What does this imply if for the income, cross-price, and own-price elasticity of demand for lemonade if the demand curve for lemonade is given by the following equation where p_l is the price of lemonade and p_c is the price of coke, and $Y$is income?

$$Q_d = Y p_l^{-\frac{1}{2}} p_c^{\frac{1}{3}}$$

Differentiation

The derivative is an important concept in mathematics. Consider the quadratic curve pictured below. We can see that as we move along the curve, its steepness changes. First, it is very large as the curve slopes downwards. As we move closer to zero, the slope decreases and eventually becomes flat. Even further to the right, the curve again becomes steeper and steeper. The derivative tells us what this gradient or slope is depending on where we are on the x-axis. More specifically, it tells us the steepness of the tangent line to our curve at a given point. For example, for the function $y = x^2$ which is illustrated below, the derivative is $\frac{dy}{dx} = 2x$. We can see that when x is very negative, the gradient is also large and negative. Moreover, at $x = 0$, the gradient, matching the diagram, is also zero. It then increases again when x is positive.



Note that the derivative can be stated in several ways: $f'(x) \equiv \frac{df}{dx}$. There are many rules of differentiation. These can be derived from the limit definition of the derivative which we leave out.

- **Power rule:** $y = ax^n \Longrightarrow \frac{dy}{dx} = anx^{n-1}$
  - $y = 4x^{\frac{1}{2}} \Longrightarrow \frac{dy}{dx} = \mathbf{2x^{-\frac{1}{2}}}$

- **Chain rule:** $y = u(f(x)) \Longrightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
  - $y = u^2; u = 2x + 1 \Longrightarrow y = (2x+1)^2 \Longrightarrow \frac{dy}{dy} = (2u) \times (2) = 4u = \mathbf{8x + 4}$

- **Product Rule:** $y = g(x)f(x) = g'(x)f(x) + f(x)'g(x)$
  - $y = x^2(1+x)^2 \Rightarrow \frac{dy}{dx} = \mathbf{2x(1+x)^2 + 2x^2(1+x)}$

- **Quotient Rule:** $y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$
  - $y = \frac{x}{1+x^2} \Rightarrow \frac{dy}{dx} = \mathbf{\frac{(1+x^2) - x(1+2x)}{(1+x^2)^2}}$

- **Exponential Rule:** $y = e^{f(x)} \Rightarrow \frac{dy}{dx} = e^{f(x)}f'(x)$
  - $y = e^{x^3} \Rightarrow \frac{dy}{dx} = \mathbf{3x^2 e^{x^3}}$

- **Logarithm Rule:** $y = \ln f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{f(x)}f'(x)$
  - $y = \ln x^2 \Rightarrow \frac{dy}{dx} = \frac{1}{x^2}2x = \mathbf{\frac{2}{x}}$

These rules are used to compute derivatives of various functions. In economics, we often want to find the stationary points of functions that correspond to maxima, minima or inflection points. In all cases, we want to find the points where $\frac{dy}{dx} = 0$. It is worth noting that that we can classify stationary points using the second derivative:

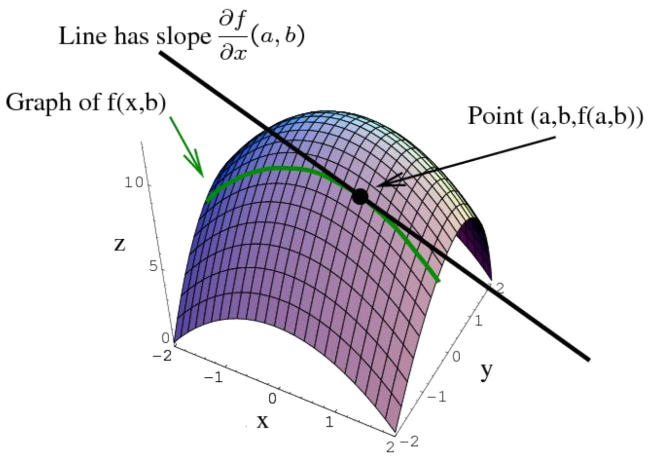
- $\frac{d^2y}{dx^2} < 0 \Rightarrow \text{Maximum}$
- $\frac{d^2y}{dx^2} > 0 \Rightarrow \text{Minimum}$
- $\frac{d^2y}{dx^2} < 0 \Rightarrow \text{Inflection Point}$

Practice Problems

1. Find the derivative of the following functions with respect to x:
    a. $f(x) = x^3$
    b. $f(x) = x^{-2}$
    c. $f(x) = 5e^{x^2}$
    d. $f(x) = \ln(2x)$
    e. $f(x) = \ln(x^3 + x^2 + 1)$
    f. $f(x) = \ln((x + 6)(e^x + 1))$
    g. $f(x) = e^x(x^2 + 1)$
    h. $f(x) = \frac{x^{\frac{1}{2}}+6}{x+1}$
    i. $f(x) = (x^2 \ln(x + 5))^{-\frac{1}{6}}$
    j. $f(x) = 2^x$
2. Find the stationary points of the function $f(x) = \frac{5x^2+4}{x}$
3. It is known that the growth of a variable over time is given by $\frac{dy}{dt}\frac{1}{y}$. Suppose that we know that $y = t^2 - t$. Work out the growth rate of y. Then, show that we can equivalently get the growth rate by taking logarithms of both sides and then differentiating with respect to time.
4. Find the second derivative for the function $y = e^{x-x^2}$

Partial Differentiation

Partial differentiation is a form of differentiation that is extended to a multivariate setting. There instead of looking at the relationship between two variables, we have a function of several variables instead. For example, compare a standard function $y = x^2$, which just has one independent variable, to $z = xy$, a function with two independent variables. When we take a derivative of a function, we are looking at the slope as we move along its curve. In a multivariate function, there are now several directions we could go in.



To visualise this idea, suppose that you are standing on the hill and can go in either of two direction: forwards and side-ways. By going either way you can trace out how steep the hill is. When we take a partial derivative, we do exactly the same thing! When taking a partial derivative with respect to x, we are just moving in the x direction and seeing how the slope of the now 3D function changes. The same goes for taking a partial derivative with respect to y. Consider the following example:

$$f(x, y) = xy + x$$

Then $\frac{\partial f}{\partial x} = y + 1$ and $\frac{\partial f}{\partial y} = x$. The reason is this is the case is that we treat the other variable as if it were a constant because we are simply moving in one direction.

Practice Problems

1. Find the partial derivative of each function with respect to x:
    a. $f(x, y) = ye^x$
    b. $f(x, y) = (x + y)^{\frac{1}{2}}$
    c. $f(x, y) = x^2 + y^2$
    d. $f(x, y) = 5xy$
    e. $f(x, y) = \frac{y^2}{x}$
    f. $f(x, y) = \frac{x^2y^3}{\sqrt{x+y}}$
    g. $f(x, y) = x^2ye^{xy}$

2. For the function $f(x, y) = x^2 + x + y^2 - 2$ find the stationary points by taking partial derivatives with respect to both variables and setting each expression equal to zero.
3. The Hessian matrix is a 2×2 table of second-order partial derivatives. Find this matrix for the function $f(x, y) = x^{\frac{1}{2}}y$ by finding $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2}{\partial y \partial x}$.

Challenge Problems

1. For the function $f(x, y) = e^{y+x+1}$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

2. For the function $f(x, y) = \ln(x + y)$, find $\frac{\partial f}{\partial x}$ and $\frac{df}{\partial x \partial y}$.

3. Young's theorem states that for any "well-defined" function, $\frac{df}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$. Verify this theorem for the function $f(x, y) = x^{\frac{1}{3}}y^{\frac{2}{3}}$.

4. The chain rule for partial differentiation $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$. Using this result, derive $\frac{df}{dt}$ for the function $f(x(t), y(t), t) = x^2 + 6y + t$, where $x(t) = t^2 - 8t$ and $y(t) = \frac{1}{t^2}$.
5. Euler's theorem states that for a homogeneous function $f(x, y)$ of degree r, it holds that $\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y = rf(x, y)$. Using the chain rule for partial derivatives and the definition of a homogenous function, prove this result.